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Alioune TOP

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Estimation Paramétriques et Tests d'Hypothèses pour des Modèles avec Plusieurs Ruptures d'un Processus de Poisson

JURY

Rapporteurs : **Delphine BLANKE,** Professeur, Université d'Avignon (France)

Gennady MARTYNOV, Directeur de Recherche, Institut des problèmes de transmission d'information(Russie)

Examinateurs : **Aboubakary DIAKHABY,** Professeur, Université Gaston Berger (Sénégal)
Abdou Ka DIONGUE, MCF-HDR, Université Gaston Berger (Sénégal)
Saïd HAMADENE, Professeur, Université du Maine (France)
Claude DEPOLLIER, Professeur Emérite, Université du Maine (France)

Directeur de Thèse : **Yury A. KUTOYANTS**, Professeur Emérite, Université du Maine (France)

Co-directeur de Thèse : **Ali Souleymane DABYE**, Professeur, Université Gaston Berger (Sénégal)

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Résumé

Ce travail est consacré aux problèmes d'estimation paramétriques, aux tests d'hypothèses et aux tests d'ajustement pour les processus de Poisson non homogènes.

Tout d'abord on a étudié deux modèles ayant chacun deux sauts localisés par un paramètre inconnu. Pour le premier modèle la somme des sauts est positive. Tandis que le second a un changement de régime et constant par morceaux. La somme de ses deux sauts est nulle. Ainsi pour chacun de ces modèles nous avons étudié les propriétés asymptotiques de l'estimateur bayésien (EB) et celui du maximum de vraisemblance(EMV). Nous avons montré la consistance, la convergence en distribution et la convergence des moments. En particulier l'estimateur bayésien est asymptotiquement efficace. Pour le second modèle nous avons aussi considéré le test d'une hypothèse simple contre une alternative unilatérale et nous avons décrit les propriétés asymptotiques (choix du seuil et puissance) du test de Wald (WT) et du test du rapport de vraisemblance généralisé (GRLT).

Les démonstrations sont basées sur la méthode d'Ibragimov et Khasminskii. Cette dernière repose sur la convergence faible du rapport de vraisemblance normalisé dans l'espace de Skorohod sous certains critères de tension des familles de mesure correspondantes.

Par des simulations numériques, les variances limites nous ont permis de conclure que l'EB est meilleur que celui du EMV. Lorsque la somme des sauts est nulle, nous avons développé une approche numérique pour le EMV.

Ensuite on a considéré le problème de construction d'un test d'ajustement pour un modèle avec un paramètre d'échelle. On a montré que dans ce cas, le test de Cramér-von Mises est asymptotiquement "parameter-free" et est consistent.
Mots clés
Estimateur bayésien, estimateur du maximum de vraisemblance, processus de Poisson non homogènes, modèle de rupture, rapport de vraisemblance, test d'hypothèse, test d'ajustement.

Abstract

This work is devoted to the parametric estimation, hypothesis testing and goodness-of-fit test problems for non homogenous Poisson processes.

First we consider two models having two jumps located by an unknown parameter. For the first model the sum of jumps is positive. The second is a model of switching intensity, piecewise constant and the sum of jumps is zero. Thus, for each model, we studied the asymptotic properties of the Bayesian estimator (BE) and the likelihood estimator (MLE). The consistency, the convergence in distribution and the convergence of moments are shown. In particular we show that the BE is asymptotically efficient. For the second model we also consider the problem of a simple hypothesis testing against a one- sided alternative. The asymptotic properties (choice of the threshold and power) of Wald test (WT) and the generalized likelihood ratio test (GRLT) are described.

For the proofs we use the method of Ibragimov and Khasminskii. This method is based on the weak convergence of the normalized likelihood ratio in the Skorohod space under some tightness criterion of the corresponding families of measure.

By numerical simulations, the limiting variances of estimators allows us to conclude that the BE outperforms the MLE. In the situation where the sum of jumps is zero, we developed a numerical approach to obtain the MLE.

Then we consider the problem of construction of goodness-of-test for a model with scale parameter. We show that the Cramér-von Mises type test is asymptotically parameter-free. It is also consistent.

Key Words.

Bayesian estimator, maximum likelihood estimator, non homogenous Poisson process, change-point model, likelihood ratio, hypothesis testing, goodness-of-fit test.

Contents

Introduction

La fréquence des changements de situation dans plusieurs domaines scientifiques explique l'intérêt porté à l'analyse statistique des points de rupture et de l'estimation. En effet, dans la pratique, lorsque les structures de contrôle révèlent qu'il y'a des ruptures quelque part dans l'évolution du phénomène étudié, il est naturel que l'on veuille localiser la position de ces ruptures qui est à l'origine du changement de régime. Ainsi, cette localisation va permettre aux décideurs de modifier le problème initial pour assurer une meilleure interprétation des données et éventuellement faire des prévisions. Du point de vue statistique, une rupture est un lieu ou un temps de sorte que les observations suivent une distribution jusqu'à ce point et suivent une autre distribution après celui-ci. Plusieurs problèmes peuvent être définis similairement. Ainsi l'approche est double: on peut se contenter de vérifier s'il y'a rupture (souvent considérer comme un problème de test d'hypothèse) ou bien de localiser le point de rupture s'il y'a lieu (vu parfois comme un problème d'estimation).

Par ailleurs les processus de Poisson modélisent des jets de points aléatoires sur des ensembles mesurables très généraux qui peuvent représenter la projection d'étoiles sur la voute céleste, les positions d'arbres dans une forêt, des sites archéologiques ou des commutateurs téléphoniques. Les points semblent être distribués au hasard dans le plan et on remarque une forme d'inhomogénéité dans leur répartition. Des situations pareilles peuvent se produire dans d'autres dimensions ou dans d'autres géométries plus compliquées. Les processus ponctuels modélisent de tels phénomènes en tant que répartitions de points au hasard et permettent de d'écrire leurs propriétés à l'aide de la théorie des probabilités. Il existe de nombreux processus ponctuels adaptés aux divers formes de modélisation. Parmi eux, nous avons les **processus d'inhibition** (par exemple processus de Gibbs) qui permettent de modéliser des phénomènes de répulsion entre des particules, les **processus de cluster** (par exemple processus de Cox) qui fournissent des modèles adaptés à l'étude des phénomènes d'attraction entre points, les **processus de Poisson** ....

Ces derniers, en plus d'être des processus ponctuels, possèdent des propriétés similaires à celles des processus de comptage d'évênements espacés par des durées indépendantes et équidistribuées de loi exponentielle. Ils modélisent des répartitions aléatoires sur $\mathbb{R}_+$. Ainsi donc les processus de Poissson non homogènes modélisent tous les processus ponctuels dont les évênements sont localisés de manière indépendante. Ils sont entièrement caractérisés par leur mesure d'intensité et sont utilisés dans beaucoup de problèmes appliqués notamment en *télécommunication optique*, en *astronomie*, en *biologie*, en *médecine*, en *analyse d'image*, en *fiabilité*.

Les premières études concernant les modèles de ruptures remontent aux années 1950. Depuis cette date jusqu'à nos jours, plusieurs articles et livres ont été publiés dans de nombreux journaux. Plusieurs d'entre eux traitent les problèmes d'estimations dans le cas de variables aléatoires identiquement distribuées. Un autre sujet très populaire de ce genre est l'étude des modèles de régression tels que la régression linéaire et l'auto regression. Dans cette même perspective, plusieurs résultats intéressants ont ét enrégistrés en théorie asymptotique des processus stochastiques avec des modèles de ruptures notamment les processus de Poisson, les processus de diffusion et les équations différentielles. Les méthodes utilisées sont souvent basées sur le rapport de vraisemblance.

Conjoncture
On sait que le nombre d'appels dans une centrale téléphonique peut être modéliser par un processus de Poisson.

Ainsi admettons les hypothèses suivantes:

- *dans une ville il y'a $k > 1$ centrales téléphoniques $C_1, C_2, \cdots, C_k$ et que le nombre d'appel dans chacune d'entre elles suit une loi de Poisson de paramètre $\lambda_1, \lambda_2, \cdots, \lambda_k$ respectivement entre 8H et 12H,*
- *de 8H à 10H toutes les k centrales fonctionnent et chacune reçoit l'appel de ses clients,*
- *de 10H à 11H, pour des raisons diverses, les centrales $C_1, C_2, \cdots, C_{k-1}$ ne fonctionnent plus et que leurs clients utilsent pendant cette durée la centrale C_k,*
- *de 11H à 12H la situation est revenue à la normale comme entre 8H et 10H.*

Ainsi dans C_k l'intensité du processus qui la caractérise est λ_k entre 8H et 10H et entre 11H et 12H. Elle vaut λ_0 entre 10H et 11H ($\lambda_0(t)$ lorsqu'elle dépend du temps) différente de λ_k. Cette différence est due à la rupture opérée au niveau des centrales $C_1, C_2, \cdots, C_{k-1}$ à partir de 10H.

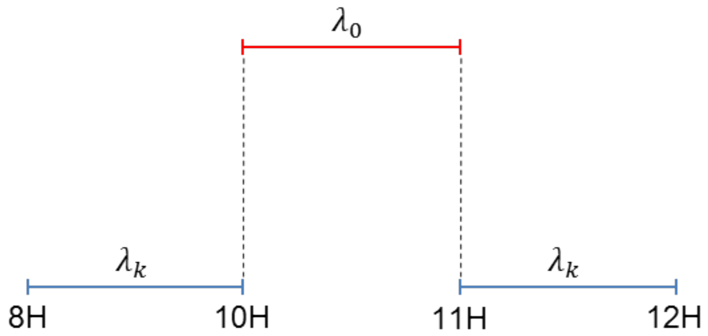

Figure 1: Schématisation de la centrale C_k

Question: Si l'instant de rupture t=10H n'est pas connu i.e. t=θ, comment l'estimer?

Cette thèse tente d'apporter entre autres des résolutions aux problématiques similaires à la question susvisée. Il faut noter aussi qu'il y'a plusieurs types de modèles de rupture et que le choix dépend en général du contexte pratique de la problématique en question. Mais il peut dépendre aussi par le fait de vouloir améliorer les propriétés des estimateurs qui sont mis en jeux. Typiquement nous allons considérer plusieurs modèles particuliers d'un processus de Poisson non homogène avec deux ruptures localisées par le paramètre inconnu θ. Ainsi, le travail repose sur l'estimation de ce paramètre. Le rapport de vraisemblance normalisé de chaque modèle converge vers une fonction exponentielle composée de la différence entre deux processus de Poisson affectés des coefficients. La pièce maitresse des méthodes utilisées est la convergence faible du rapport de vraisemblance normalisé dans un espace métrique adapté. En particulier, nous allons vérifier la convergence des distributions finies dimensionnelles et les critères de tension des familles de mesures correspondantes dans la métrique de Skorohod. Ces résultats nous permettront de prouver que l'estimateur bayésien et celui du maximum de vraisemblance construits à partir de n réalisations d'un processus de Poisson, sont consistants , convergent en loi et leurs moments aussi convergent avec une vitesse égale à n. Des simulations seront proposées pour illustrer les résultats théoriques obtenus et comparer les performances de l'estimateur bayesien par rapport à celles de l'estimateur du maximum de vraisemblance.

Pour le second modèle (deux ruptures dont la somme des sauts est nulle), nous avons considéré les tests d'hypothèses avec une alternative unilatérale. Ainsi nous avons étudié les propriétés asymptotiques (optimalité pour une classe de niveau asymptotique fixée et puissance suivant le critère de Neyman-Pearson) des tests du rapport de vraisemblance (GRLT) basé sur le maximum de la fonction de vraisemblance et le test de Wald (WT) basé sur l'estimateur du maximum de vraisemblance. Les propriétés de ces tests dépendent entièrement des propriétés asymptotiques du rapport de vraisemblance normalisé.

Toutes les démonstrations reposent sur les méthodes d'Ibragimov-Khasminski [32] dans le cadre de l'estimation de la densité discontinue de variables et celles de Kutoyants [41] dans le cadre de l'estimation paramétrique pour un modèle d'intensité d'un processus de Poisson non homogène.

Par ailleurs on a étudié les tests d'ajustements d'hypothèses paramétriques composées pour la statistique de Cramér-von Mises. Pour un modèle avec un paramètre d'échelle, nous avons proposé un test de type Cramér-von Mises qui est asymptotiquement "parameter free" i.e. la statistique limite ne dépend pas du paramètre inclus dans le modèle. Ce test est aussi consistent.

Voici le résumé des différents résultats obtenus dans cette thèse.

0.1 Estimation paramétriques pour les processus de Poisson: Cas d'un modèle avec deux ruptures dont la somme est nulle

Dans le Chapitre 2 on étudie un modèle ayant deux sauts localisés par un paramètre inconnu et dont la somme des sauts est positive.

On suppose que les observations $X^{(n)} = (X_1, \ldots, X_n)$ sont des processus de Poisson non homogènes indépendants $X_j = \{X_j(t),\, 0 \leq t \leq T\}$, $j = 1, \ldots, n$ avec la même fonction d'intensité

$$\lambda(\theta, t) = \lambda_0 + \lambda_1(t)\mathbb{1}_{\{\theta \leq t \leq \theta + \tau_0\}}, \quad 0 \leq t \leq \tau.$$

Ici

$$\theta \in \Theta = (\alpha, \beta), \quad \tau = T - \tau_0, \quad 0 < \alpha < \beta < \beta + \tau_0 < \tau.$$

Sous cette condition la fonction d'intensité admet deux sauts sur l'intervalle des observations et ceci pour tout $\theta \in \Theta$. Rappelons que

$$\mathbf{E}_\theta X_j(t) = \Lambda(\theta, t) = \int_0^t \lambda(\theta, s)\, ds.$$

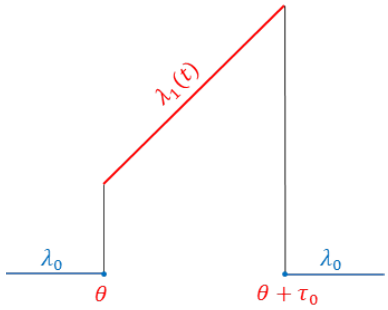

Figure 2: comportement de la fonction d'intensité $\lambda(\theta, t)$

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Ce modèle est souvent utilisé en statistique de la radiophysique: la fonction $\lambda_1(t)\mathbb{1}_{\{\theta\leq t\leq\theta+\tau_0\}}$ est un signal de longueur τ_0 et $\lambda_0>0$ est un bruit Poissonnien. Il est aussi utilisé en télécommunication optique: le paramètre (l'information) θ est transmis à travers un canal optique avec une intensité $\lambda_1(t)\mathbb{1}_{\{\theta\leq t\leq\theta+\tau_0\}}$ et λ_0 représente l'intensité du bruit.

Le paramètre θ est supposé être inconnu et nous devons l'estimer par les observations X^n. Ainsi on s'intéresse au comportement asymptotique ($n\to\infty$) de l'estimateur bayésien (BE) et celui du maximum de vraisemblance (MLE).

Notons par $\mathbf{P}_\theta^{(n)}$ la mesure induite dans l'espace des observations de n réalisations d'un processus de Poisson de fonction d'intensité $\lambda(\theta, t)$, $0\leq t\leq\tau$. Comme $\lambda_0>0$ et $\lambda_1(t)$ est bornée les mesures $\mathbf{P}_\theta^{(n)}$ sont équivalentes et la fonction de rapport de vraisemblance

$$L\left(\theta,\theta_1,X^{(n)}\right)=\frac{d\mathbf{P}_\theta^{(n)}}{d\mathbf{P}_{\theta_1}^{(n)}}(X^{(n)})$$

est

$$\begin{aligned} L\left(\theta,\theta_1,X^{(n)}\right) &= \exp\Bigg\{\sum_{j=1}^n\int_0^\tau \ln\Big(\frac{\lambda_0+\lambda_1(t)\mathbb{1}_{\{\theta\leq t\leq\theta+\tau_0\}}}{\lambda_0+\lambda_1(t)\mathbb{1}_{\{\theta_1\leq t\leq\theta_1+\tau_0\}}}\Big)dX_j(t) \\ &\quad -n\int_0^\tau\left(\lambda_1(t)\mathbb{1}_{\{\theta\leq t\leq\theta+\tau_0\}}-\lambda_1(t)\mathbb{1}_{\{\theta_1\leq t\leq\theta_1+\tau_0\}}\right)dt\Bigg\}. \end{aligned}$$

Puisque le rapport de vraisemblance $L\left(\theta,\theta_1,X^{(n)}\right)$ est une fonction discontinue de θ, nous définissons le MLE $\hat\theta_n$ comme étant la solution de l'équation suivante:

$$\max\left\{L\left(\hat\theta_n+,\theta_1,X^{(n)}\right), L\left(\hat\theta_n-,\theta_1,X^{(n)}\right)\right\}=\sup_{\theta\in\Theta}L\left(\theta,\theta_1,X^{(n)}\right).$$

Ici θ_1 est une certaine valeur fixée et $L\left(\hat{\theta}_n+, \theta_1, X^{(n)}\right)$, $L\left(\hat{\theta}_n-, \theta_1, X^{(n)}\right)$ sont respectivement les limites à gauche et à droite de la fonction $L\left(\theta, \theta_1, X^{(n)}\right)$ au point $\hat{\theta}_n$.

Pour introduire l'estimateur bayésien, nous supposons que le paramètre inconnu est une variable aléatoire avec une densité $p(\theta)$ $\theta \in \Theta$, connue, positive et continue. Ainsi le BE $\widetilde{\theta_n}$ est une espérance conditionnelle qui peut être écrite comme suit:

$$\widetilde{\theta_n} = \mathbf{E}\left(\theta / X^{(n)}\right) = \int_\alpha^\beta \theta p(\theta) L\left(\theta, X^{(n)}\right)\, d\theta \left(\int_\alpha^\beta p(\theta) L\left(\theta, X^{(n)}\right)\, d\theta \right)^{-1}.$$

Pour décrire les propriétés des estimateurs nous avons besoin des notations suivantes:

$$Z_{\theta_0}(u) = \begin{cases} \exp\left\{ \rho_1\, X^+(u) + \rho_2\, Y^+(u) - ru \right\} & u \geq 0 \\ \exp\left\{ -\rho_1\, X^-(-u) - \rho_2\, Y^-(-u) - ru \right\} & u < 0, \end{cases}$$

où $X^+(\cdot)$, $X^-(\cdot)$, $Y^+(\cdot)$ et $Y^-(\cdot)$ sont des processus de Poisson sur $\mathbb{R}_+$ d'intensités respectives $\lambda_0 + \lambda_1(\theta_0)$, λ_0, λ_0 et $\lambda_0 + \lambda_1(\theta_0 + \tau_0)$ et

$$\rho_1 = \ln \frac{\lambda_0}{\lambda_0 + \lambda_1(\theta_0)}, \quad \rho_2 = \ln \frac{\lambda_0 + \lambda_1(\theta_0 + \tau_0)}{\lambda_0}, \quad r = \lambda_1(\theta_0 + \tau_0) - \lambda_1(\theta_0).$$

Notons $u = \frac{v}{r}$, $X_1^\pm(v) = X^\pm(\frac{v}{r})$ et $Y_1^\pm(v) = Y^\pm(\frac{v}{r})$. Ainsi par un changement de temps linéaire , nous obtenons

$$Z_\rho^*(v) := \begin{cases} \exp\left\{ \rho_1 X_1^+(v) + \rho_2 Y_1^+(v) - v \right\} & v \geq 0 \\ \exp\left\{ -\rho_1 X_1^-(-v) - \rho_2 Y_1^-(-v) - v \right\} & v < 0 \end{cases} \tag{1}$$

où $X_1^+(\cdot)$, $X_1^-(\cdot)$, $Y_1^+(\cdot)$ et $Y_1^-(\cdot)$ sont des processus de Poisson indépendants sur $\mathbb{R}_+$ d'intensités respectives $\frac{\lambda_0}{re^{\rho_1}}$, $\frac{\lambda_0}{r}$, $\frac{\lambda_0}{r}$ et $\frac{\lambda_0 e^{\rho_2}}{r}$.

Introduisons les variables aléatoires $\widetilde{u}$, $\hat{u}$ $\widetilde{u}_\rho$ et $\hat{u}_\rho$ définies par

$$\widetilde{u} = \int_{-\infty}^{+\infty} uZ_{\theta_0}(u)\,du \left(\int_{-\infty}^{+\infty} Z_{\theta_0}(u)\,du \right)^{-1},$$

$$\max\left\{Z_{\theta_0}(\hat{u}-), Z_{\theta_0}(\hat{u}+)\right\} = \sup_{u\in\mathbb{R}} Z_{\theta_0}(u)$$

$$\widetilde{u}_\rho = \int_{-\infty}^{+\infty} vZ_\rho^*(v)\,dv \left(\int_{-\infty}^{+\infty} Z_\rho^*(v)\,dv \right)^{-1}$$

et

$$\max\left\{Z_\rho^*(\hat{u}_\rho-), Z_\rho^*(\hat{u}_\rho+)\right\} = \sup_{v\in\mathbb{R}} Z_\rho^*(v).$$

Nous avons également $\hat{u} \equiv \frac{\hat{u}_\rho}{r}$ et $\widetilde{u} \equiv \frac{\widetilde{u}_\rho}{r}$. Notre objectif maintenant est de trouver les propriétés asymptotiques des estimateurs $\widetilde{\theta}_n$ et $\hat{\theta}_n$ de θ lorsque $n \longrightarrow +\infty$.

Introduisons la condition C_0

1. Les constantes λ_0 and τ_0 sont connues et strictement positives.
2. La fonction $\lambda_1(\cdot)$ est strictement positive, strictement croissante et continue pour tout $t \in [0, \tau]$.

Les principaux résultats de cette partie sont les suivants.

Tout d'abord, on établit une borne inférieure du risque moyenne quadratique pour tous les estimateurs de ce modèle.

Si la condition C_0 est satisfaite, alors

$$\lim_{\delta\to 0} \varliminf_{n\to+\infty} \inf_{\overline{\theta}_n} \sup_{|\theta-\theta_0|<\delta} n^2 \mathbf{E}_\theta \left(\overline{\theta}_n - \theta\right)^2 \geq \mathbf{E}_{\theta_0} \widetilde{u}^2 = \frac{\mathbf{E}_{\theta_0}\left(\widetilde{u}_\rho^2\right)}{r^2} \tag{2}$$

où la borne inf est prise sur tous les estimateurs possibles $\overline{\theta}_n$ du paramètre θ.

L'inégalité (2) nous permet d'introduire un estimateur asymptotiquement efficace pour ce problème.

On dit que l' estimateur $\overline{\theta}_n$ est *asymptotiquement efficace* si pour tout $\theta_0 \in \Theta$ nous avons

$$\lim_{\delta\to 0} \lim_{n\to+\infty} \sup_{|\theta-\theta_0|<\delta} n^2 \mathbf{E}_\theta \left(\overline{\theta}_n - \theta\right)^2 = \frac{\mathbf{E}_{\theta_0}\left(\widetilde{u}_\rho^2\right)}{r^2}$$

Le premier estimateur étudié est l'estimateur baysien $\widetilde{\theta}_n$ et ses propriétés sont décrites comme suit

Si la condition C_0 est satisfaite, alors l'estimateur Bayésien $\widetilde{\theta}_n$ vérifie uniformément en $\theta_0 \in \mathbf{K}$ ($\mathbf{K}$ un compact dans Θ) les relations:

la consistence

$$\mathbf{P}_{\theta_0} - \lim_{n\to+\infty} \widetilde{\theta}_n = \theta_0,$$

la convergence en loi

$$\mathcal{L}_\theta\left\{n\left(\widetilde{\theta}_n - \theta_0\right)\right\} \Rightarrow \mathcal{L}\left(\frac{\widetilde{u}_\rho}{r}\right)$$

et le polynôme des moments converge

$$\lim_{n\to+\infty} \mathbf{E}_{\theta_0} |n\left(\widetilde{\theta}_n - \theta_0\right)|^p = \mathbf{E}_{\theta_0} \frac{|\widetilde{u}_\rho|^p}{|r|^p}$$

pour tout $p > 0$. Il est aussi asymptotiquement efficace.

Le second est l'estimateur du maximum de vraisemblance et ses propriétés sont les suivantes:

Si la condition C_0 est satisfaite, alors l'estimateur du maximum de vraisemblance $\hat{\theta}_n$ vérifie uniformément en $\theta_0 \in \mathbf{K}$ les relations:

la consistance

$$\mathbf{P}_{\theta_0} - \lim_{n\to+\infty} \hat{\theta}_n = \theta_0,$$

la convergence en loi

$$\mathcal{L}_\theta\left\{n\left(\hat{\theta}_n - \theta_0\right)\right\} \Rightarrow \mathcal{L}\left(\frac{\hat{u}_\rho}{r}\right)$$

et le polynôme des moments converge

$$\lim_{n\to+\infty} \mathbf{E}_{\theta_0} |n\left(\hat{\theta}_n - \theta_0\right)|^p = \frac{\mathbf{E}_{\theta_0} |\hat{u}_\rho|^p}{|r|^p}$$

pour tout $p > 0$. Les simulations nous montrent que dans ce cadre l'estimateur bayésien est meilleur que celui du maximum de vraisemblance car pour les moyennes empiriques basées sur $N = 10^4$ réplications, nous avons la relation suivante

$$\sigma^2_{MLE} = \frac{1}{N}\sum_{l=1}^{N} \hat{u}_l^2 > \sigma^2_{BE} = \frac{1}{N}\sum_{l=1}^{N} \widetilde{u}_l^2.$$

0.2 Un modèle avec deux sauts dont la somme est nulle

Dans le Chapitre 3 nous avons considéré un modèle avec un changement de régime et constant par morceaux ayant deux sauts localisés par un paramètre inconnu. La somme des sauts est nulle. Ainsi on se donne n réalisations de processus de Poisson indépendants $X^{(n)} = (X_1, \ldots, X_n)$ où les $X_j = \left\{X_j(t),\, 0 \leq t \leq T\right\}$, j= $1, \ldots, n$ des processus de Poisson

$$\mathbf{E}_\theta X_j(t) = \Lambda\left(\theta, t\right) = \int_0^t \lambda\left(\theta, s\right)\, ds.$$

La fonction d'intensité est

$$\lambda\left(\theta, t\right) = \lambda_0 + \lambda_1 1\!\!\mathrm{I}_{\{\theta \leq t \leq \theta + \tau_0\}}$$

où

$$\theta \in \Theta = (\alpha, \beta), \quad \tau = T - \tau_0, \quad 0 < \alpha < \beta < \beta + \tau_0 < \tau.$$

L'importance de ce modèle réside au fait que la résultante des sauts est nulle; ce qui laisse entendre que les deux sauts sont en opposition de phase. Par conséquent, on note une amélioration des qualités des estimateurs offrant ainsi des résultants plus intéressants en pratique.

0.2.1 Estimation paramétriques

Le paramètre θ aussi bien que $\theta + \tau_0$ correspond chacun à la localisation d'un saut dans la fonction d'intensité $\lambda\left(\theta, t\right)$. Ainsi, les instants de sauts sont inconnus et nous devons estimer la vraie valeur correspondant au paramètre θ. La vraisemblance du modèle est donnée par

$$L\left(\theta, X^{(n)}\right) = \exp\Bigg\{\sum_{j=1}^{n} \int_0^{\tau} \ln\bigl(\lambda_0 + \lambda_1 1\!\!\mathrm{I}_{\{\theta \leq t \leq \theta + \tau_0\}}\bigr) dX_j(t)$$
$$-n \int_0^{\tau} \bigl(\lambda_0 + \lambda_1 1\!\!\mathrm{I}_{\{\theta \leq t \leq \theta + \tau_0\}} - 1\bigr) dt\Bigg\}.$$

Pour décrire les propriétés des estimateurs nous avons besoin des notations suivantes:

Introduidsons le processus

$$Z(v) := \begin{cases} \exp\left\{\rho\left(Y^{+}(u) - X^{+}(u)\right)\right\} & u \geq 0 \\ \exp\left\{\rho\left(Y^{-}(u) - X^{-}(u)\right))\right\} & u < 0 \end{cases}$$

où $\rho = \ln \frac{\lambda_0+\lambda_1}{\lambda_0}$. Les processus de Poisson $X^{+}(\cdot)$, $X^{-}(\cdot)$, $Y^{+}(\cdot)$ et $Y^{-}(\cdot)$ sont indépendants $\mathbb{R}_+$ d'intensités respectives $\lambda_0 + \lambda_1$, λ_0, λ_0 et $\lambda_0 + \lambda_1$.

Soient $\widetilde{u}$, $\hat{u}$ des variables aléatoires telles que :

$$\widetilde{u} = \int_{-\infty}^{+\infty} uZ(u)\, du \left(\int_{-\infty}^{+\infty} Z(u)\, du\right)^{-1}, \quad Z(\hat{u}) = \sup_{u\in\mathbb{R}} Z(u)$$

où $\hat{u}_l < \hat{u} < \hat{u}_r$. L'intervalle $[\hat{u}_l, \hat{u}_r]$ est l'intervalle le plus élevé $Z(u)$.

Admettons la condition $\mathbf{B_0}$

- **a.** *La constantes λ_0 et λ_1 sont t connues et strictement positives.*

Les résultats obtenus dans cette partie sont les suivants:

$$\lim_{\delta\to 0} \varliminf_{n\to+\infty} \inf_{\overline{\theta}_n} \sup_{|\theta-\theta_0|<\delta} n^2 \mathbf{E}_\theta \left(\overline{\theta}_n - \theta\right)^2 \geq \mathbf{E}_{\theta_0} \widetilde{u}^2 = \mathbf{E}_{\theta_0} \left(\widetilde{u}^2\right) \tag{3}$$

où la borne inf est prise sur tous les estimateurs possibles $\overline{\theta}_n$ du paramètre θ.

On dit que l' estimateur $\overline{\theta}_n$ est *asymptotiquement efficace* si pour tout $\theta_0 \in \Theta$ nous avons

$$\lim_{\delta\to 0} \lim_{n\to+\infty} \sup_{|\theta-\theta_0|<\delta} n^2 \mathbf{E}_\theta \left(\overline{\theta}_n - \theta\right)^2 = \mathbf{E}_{\theta_0} \left(\widetilde{u}^2\right)$$

L'estimateur baysien $\widetilde{\theta}_n$:

Si la condition $\mathbf{B_0}$ est satisfaite, alors l'estimateur bayésien $\widetilde{\theta}_n$ vérifie uniformément en $\theta_0 \in \mathbf{K}$ ($\mathbf{K}$ un compact dans Θ) les relations:

la consistance

$$\mathbf{P}_{\theta_0} - \lim_{n\to+\infty} \widetilde{\theta}_n = \theta_0,$$

la convergence en loi

$$\mathcal{L}_\theta\left\{n\left(\widetilde{\theta}_n - \theta_0\right)\right\} \Rightarrow \mathcal{L}\left(\widetilde{u}\right)$$

et le polynôme des moments converge

$$\lim_{n\to+\infty} \mathbf{E}_{\theta_0}|n\left(\widetilde{\theta}_n-\theta_0\right)|^p=\mathbf{E}_{\theta_0}|\widetilde{u}|^p$$

pour tout $p>0$. Il est aussi asymptotiquement efficace.

l'estimateur du maximum de vraisemblance

Si la condition $\mathbf{B_0}$ est satisfaite, alors l'estimateur du maximum de vraisemblance $\hat{\theta}_n$ vérifie uniformément en $\theta_0\in\mathbf{K}$ les relations:

$$\mathbf{P}_{\theta_0}-\lim_{n\to+\infty}\hat{\theta}_n=\theta_0,$$

$$\mathcal{L}_\theta\left\{n\left(\hat{\theta}_n-\theta_0\right)\right\}\Rightarrow\mathcal{L}\left(\hat{u}\right)$$

$$\lim_{n\to+\infty} \mathbf{E}_{\theta_0}|n\left(\hat{\theta}_n-\theta_0\right)|^p=\mathbf{E}_{\theta_0}|\hat{u}|^p$$

pour tout $p>0$.

0.2.2 Tests d'hypothèses

Dans cette partie, on reconduit le même modèle d'observation et on s'intéresse au schéma de test suivant

$$\mathscr{H}_0 \quad : \quad \theta=\theta_1,$$

et l'alternative

$$\mathscr{H}_1 \quad : \quad \theta>\theta_1,$$

où $\theta\in\Theta=[\theta_1,\beta)$. On considère ainsi comme alternatives une suite de modèles statistiques indexée par n et on utilise le changement de variable pour le paramètre $\theta=\theta_1+\frac{u}{n}$ où $u\in U_n=[0,n(\beta-\theta_1)]$. Le problème initial devient

$$\mathscr{H}_0 \quad : \quad u=0,$$

$$\mathscr{H}_1 \quad : \quad u>0.$$

Fixons $\varepsilon\in[0,1]$. On note $\mathcal{K}_\varepsilon$ la classe des fonctions $\overline{\Psi}_n(X^n)$ de niveau asymptotique ε i.e.

$$\mathcal{K}_\varepsilon=\left\{\overline{\Psi}_n:\lim_{n\to\infty}\mathbf{E}_{\theta_1}\overline{\Psi}_n=\varepsilon\right\};$$

et la fonction de puissance β_n de la statistique de test est

$$\beta\left(\overline{\Psi}_n, u\right) = \mathbf{E}_{\theta_u}\left(\overline{\Psi}_n\right), \quad \theta_u = \theta_1 + \frac{u}{n}$$

où $\mathbf{E}_{\theta_1}$ and $\mathbf{E}_{\theta_u}$ sont respectivement les espérances mathématiques sous les hypothèses $\mathscr{H}_0$ and $\mathscr{H}_1$. L'estimateur du maximum de vraisemblance $\hat{\theta}_n$ est définie comme une solution de l'équation

$$L\left(\hat{\theta}_n, \theta_1, X^{(n)}\right) = \sup_{\theta\in[\theta_1,\beta)} L\left(\theta, \theta_1, X^{(n)}\right).$$

Par ailleurs pour $u, u_* > 0$, on introduit $X(\cdot)$, $Y(\cdot)$, $X^*(\cdot)$ et $Y^*(\cdot)$ quatre processus de Poisson indépendants tels que

$$\mathbf{E}X(u) = (\lambda_0 + \lambda_1)u, \quad \mathbf{E}Y(u) = \lambda_0 u$$

$$\mathbf{E}X^*(u_*) = \lambda_0 u_* \quad \mathbf{E}Y^*(u_*) = (\lambda_0 + \lambda_1)u_*.$$

Définissons les processus aléatoires

$$Z(u) = \exp\left\{\rho\left(Y(u) - X(u)\right)\right\},$$

$$Z^*(u_*) = \exp\left\{\rho\left(Y^*(u_*) - X^*(u_*)\right)\right\}$$

et

$$\widetilde{Z}(u, u_*) = \frac{Z^*(u_*)}{Z^*(u)} 1\!\!1_{\{u\leq u_*\}} + \frac{Z(u)}{Z(u_*)} 1\!\!1_{\{u>u_*\}}.$$

On pose

$$\hat{G}_n = L\left(\hat{\theta}_n, \theta_1, X^{(n)}\right) = \sup_{\theta\in[\theta_1,\beta)} L\left(\theta, \theta_1, X^{(n)}\right).$$

Ainsi le test du rapport de vraisemblance généralisé (GRLT) est donné par la fonction de descision suivante

$$\overline{\Psi}_n = 1\!\!1_{\{\hat{G}_n > c_\varepsilon\}}$$

et le test de Wald (WT) par celle-ci

$$\hat{\Psi}_n = 1\!\!1_{\{n(\hat{\theta}_n - \theta_1) > b_\varepsilon\}}.$$

Les seuils c_ε and b_ε sont choisis respectivement selon la condition $\overline{\Psi}_n \in \mathcal{K}_\varepsilon$ et $\hat{\Psi}_n \in \mathcal{K}_\varepsilon$. Ainsi les résultats principaux sont les suivants:

Test de GRLT:

Supposons que la valeur c_ε est une solution de l'équation

$$\mathbf{P}\{\sup_{u>0} Z(u) > c_\varepsilon\} = \varepsilon.$$

Alors le test

$$\overline{\Psi}_n \in \mathcal{K}_\varepsilon$$

et sa fonction de puissance converge

$$\beta\left(\overline{\Psi}_n, u_*\right) \longrightarrow \mathbf{P}\{\sup_{u>0} Z^*(u_*)\widetilde{Z}(u, u_*) > c_\varepsilon\}.$$

Test de Wald:

Supposons que la valeur b_ε est une solution de l'équation

$$\mathbf{P}\{\hat{u} > b_\varepsilon\} = \varepsilon.$$

Alors le test

$$\hat{\Psi}_n \in \mathcal{K}_\varepsilon$$

et sa fonction de puissance converge

$$\beta\left(\hat{\Psi}_n, u_*\right) \longrightarrow \mathbf{P}\{\hat{u}_{u_*} > b_\varepsilon\}.$$

Les variables aléatoires $\hat{u}$ et $\hat{u}_{u_*}$ sont telles que

$$Z(\hat{u}) = \sup_{u\in\mathbb{R}^+} Z(u), \quad \widetilde{Z}(\hat{u}_{u_*}, u_*) = \sup_{u\in\mathbb{R}^+} \widetilde{Z}(u, u_*).$$

0.3 Test de type Cramér-von Mises pour un processus de Poisson non homogène avec un paramètre d'échelle

Dans le Chapitre 4, nous présentons un test d'ajustement asymptotiquement "parameter-free" et consistent dans le cas d'un processus de Poisson non homogènes. L'hypothèse de base est paramétrique composée. La statistique de Cramér-von Mises avec le paramètre remplacé par l'estimateur du maximum de vraisemblance est considérée. Ainsi nous montrons que dans le cas d'un paramètre d'échelle, la distribution limite de la statistique de test ne dépend pas du paramètre inconnu.

Introduisons la fonction d'intensité

$$\Lambda_0(t,\theta)=\theta\int_{-\infty}^{t}\lambda_0\left(\frac{s}{\theta}\right)\,\frac{ds}{\theta}=\theta\Lambda_0\left(\frac{t}{\theta}\right).$$

et une famille paramétrique

$$\mathcal{L}(\Theta)=\left\{\Lambda_0(t,\theta)=\theta\Lambda_0\left(\frac{t}{\theta}\right),\theta\in\Theta=(\alpha,\ \beta)\right\}$$

où $\Lambda_0(\cdot)$ est une certaine fonction croissante avec les propriétés suivantes:

$$\Lambda_0(-\infty)=0,\quad \Lambda_0(\infty)<\infty,\quad \Lambda_0(t)=\int_{-\infty}^{t}\lambda_0(s)\ ds.$$

Nous observons n processus de Poisson non homogènes indépendants $X^{(n)}=(X_1,\ldots,X_n),\ X_j=\left\{X_j(t),\ t\in\mathbb{R}\right\}$ avec la même fonction moyenne $\Lambda(\cdot)$. Le paramètre inconnu θ est remplacé par son estimateur du maximum de vraisemblance $\hat{\theta}_n$ et notre statistique est définie comme suit:

$$\Delta_n=\frac{n}{\hat{\theta}_n^2}\int_{-\infty}^{+\infty}\left[\widehat{\Lambda}_n(t)-\Lambda_0\left(t,\hat{\theta}_n\right)\right]^2\lambda_0\left(\frac{t}{\hat{\theta}_n}\right)\ dt=\frac{\widetilde{\Delta}_n}{\hat{\theta}_n^2}.$$

Ainsi le test de type Cramér-von Mises est:

$$\hat{\Psi}_n(X^n)=\mathbb{1}_{\{\Delta_n>c_\varepsilon\}}.$$

Le seuil c_ε doit être choisi de sorte que ce test appartiendra à la classe des tests de niveau asymptotique ε

$$\mathcal{K}_\varepsilon=\left\{\overline{\Psi}_n:\lim_{n\to\infty}\mathbf{E}_\theta\overline{\Psi}_n=\varepsilon,\quad \theta\in\Theta\right\}.$$

Comme nous utilisons le MLE $\hat{\theta}_n$, nous avons besoin des conditions de régularité. Supposons que la fonction d'intensité $\lambda_0(\cdot)$ est strictement positive et suffisamment régulière. Sous ces conditions le MLE est asymptotiquement normal et le polynôme des moments converge (voir [41]).

Celà étant, nous devons tester l'hypothèse paramétrique composée

$$\mathscr{H}_0\quad:\qquad\Lambda(\cdot)\in\mathcal{L}(\Theta)$$

contre l'alternative

$$\mathscr{H}_1\quad:\qquad\Lambda(\cdot)\notin\mathcal{L}(\Theta).$$

Plus précisément nous supposons sous l'alternative que

$$\inf_{\theta\in\Theta} \|\Lambda(\cdot) - \Lambda_0(\cdot,\theta)\| > 0.$$

Ici et dans tout le reste du travail la notation $\|\cdot\|$ est la norme L_2 suivante

$$\|f\|_\theta^2 = \int_{-\infty}^{\infty} f(t)^2 \lambda_0\left(\frac{t}{\theta}\right) dt.$$

Nous montrerons que pour de telles alternatives le test est consistant. Il sera aussi consistant pour une autre classe d'alternative

$$\mathscr{H}_1^\rho \quad : \quad \Lambda(\cdot) \in \mathcal{F}_\rho = \left\{\Lambda(\cdot) : \inf_{\theta\in\Theta} \|\Lambda(\cdot) - \Lambda_0(\cdot,\theta)\|_\theta > \rho\right\}.$$

Ici $\rho > 0$ est un nombre donné. Nous supposons aussi que $\mathcal{F}_\rho$ est telle que

$$\sup_{\Lambda\in\mathcal{F}_\rho} \Lambda(\infty) < \infty.$$

Introduisons la variable aléatoire suivante:

$$\Delta_0 = \int_{-\infty}^{\infty} \left[W(\Lambda_0(t)) + (\Lambda_0(t) - t\lambda_0(t)) \int_{-\infty}^{\infty} \mathrm{I}_0{}^{-1} s \frac{\dot\lambda_0(s)}{\lambda_0(s)} dW(\Lambda_0(s))\right]^2 d\Lambda_0(t)$$

où $W(\cdot)$ est un processus de Wiener standard. La constante c_ε est la solution de l'équation

$$\mathbf{P}\{\Delta_0 > c_\varepsilon\} = \varepsilon.$$

Conditions A.

- **a1.** *La fonction* $\sqrt{\lambda_0(\cdot)} \in \mathcal{L}_2(\mathbb{R})$ *est strictement positive et trois fois continument différentiable.*
- **a2.** *Ces dérivées appartiennent à l'espace* $\mathcal{L}_2(\mathbb{R})$ *.*
  L'information de Fisher

$$\mathrm{I}(\theta) = \frac{1}{\theta} \int_{-\infty}^{+\infty} t^2 \frac{\dot\lambda_0^2(t)}{\lambda_0(t)} dt > 0.$$

- **a3.** *La condition* $\dot\lambda_0(\cdot) \in \mathcal{L}_1(\mathbb{R})$ *.*

A la lumière de tout ce qui précède nous avons les résultats principaux suivants:

- Sous la condition **A** nous avons le test

$$\hat\Psi_n = 1\!\!\mathrm{I}_{\{\Delta_n > c_\varepsilon\}}$$

appartient à la classe $\mathcal{K}_\varepsilon$ c'est-à-dire le test de Cramér-von Mises avec un paramètre d'échelle est asymptotiquement de niveau ε. Ainsi pour tout $\varepsilon > 0$, on peut calculer c_ε.

- Sous la condition **A** le test

$$\hat{\Psi}_n = 1\!\!1_{\{\Delta_n > c_\varepsilon\}}$$

est consistant sous l'alternative $\mathscr{H}_1$, c'est-à-dire, pour tout $\Lambda \notin \mathcal{L}(\Theta)$ nous avons:

$$\beta\left(\hat{\Psi}_n, \Lambda\right) \underset{n\to\infty}{\longrightarrow} 1,$$

et il est uniformément consistant sous les alternatives $\mathscr{H}_1^{\rho}$,c'est-à-dire,

$$\inf_{\Lambda(\cdot)\in\mathcal{F}_\rho} \beta\left(\hat{\Psi}_n, \Lambda\right) \underset{n\to\infty}{\longrightarrow} 1.$$

Les résultats de ces chapitres ont fait l'objet de publications et de présentations orales.

Articles:

1. Dabye, A.S., Tanguep, D.W. and Top, A., On the Cramér-von Mises test for Poisson process with scale parameter. *Far East Journal of Theoritical statistics*, accepté avec "minors revision".
2. Chernoyarov, O.V., Kutoyants, Yu.A. and Top, A., On multiple change point estimation for Poisson process: case of non zero jumps sum. *Soumis pour publication.*
3. Dabye, A.S., Dachian, S., and Top, A., On multiple change point estimation for Poisson process: case of zero jumps sum. *Soumis pour publication.*
4. Dabye, A.S., Tanguep, D.W. and Top, A., On Asymptotically Parameter free test of the Kolmogorov-Smirnov type stattistic for Poisson process.*Soumis pour publication.*"non inclus dans la thèse"

Posters et Communications orales:

1. On the Cramér-von Mises test for Poisson process with scale parameter. *Conférence en Probabilités et Statistiques à l'Université Gaston Berger de Saint-louis (Sénégal), mars 2014*
2. Dabye, A.S., Tanguep, D.W. and Top, A., On Asymptotically Parameter free test of the Kolmogorov-Smirnov type stattistic for Poisson process. *"Dakar International Conference on Recent Development in Applied Statistics "(DICDAS) à l'université Cheikh Anta Diop de Dakar (Sénégal), mars 2014.*

3. On the Cramér-von Mises test for Poisson process with scale parameter. *Congrès de l'Agence Universitaire de la Francophonie (AUF) sur les sciences fondamentales (Mathématiques ,Informatiques) à Antananarivo (Madagascar), octobre 2014.*

4. On multiple change point estimation for Poisson process: case of non zero jumps sum. *Conférence internationale sur la théorie asymptotique des processus stochastiques, Le Mans (France), mars 2015.*

5. On multiple change point estimation for Poisson process: case of non zero jumps sum. *Congrès de l'Agence Universitaire de la Francophonie (AUF) sur les sciences fondamentales (Mathématiques ,Informatiques), à Cotonou, (Benin), octobre 2015.*

6. On multiple change point estimation for Poisson process: case of non zero jumps sum. *Statistical Methods For Dynamical Stochastic Models, DYNSTOCH 2016 June at Rennes (France).*

Chapter 1

Auxiliary results

1.1 Introduction

Dans ce chapitre, nous commençerons par étudier les processus de Poisson non homogènes. Nous donnerons ensuite les définitions et propriétés liées à l'intégrale stochastique et au rapport de vraisemblance au sens des processus de Poisson. Ces résultats seront utilisés dans les chapitres suivants. Enfin, nous ferons un rappel de certains resultats déjà obtenus sur l'estimation paramétrique des processus stochastiques dans le cas regulier comme dans le cas singulier et des tests d'ajustements.

1.2 Processus de Poisson non homogènes

Processus ponctuel et fonction aléatoire de comptage.

Un processus ponctuel sur [0,T] se décrit, pour entier M, par la donnée d'une suite croissante de points aléatoires $0 < t_1 < t_2 < \cdots < t_M \leq \cdots$ dans [0,T] qui sont des variables aléatoires définies sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbf{P})$.

Posons

$$\begin{aligned} s_1 &= t_1 \\ s_2 &= t_2 - t_1 \\ &\dots \\ s_M &= t_M - t_{M-1}, \\ &\dots \end{aligned}$$

$t_0 = 0$ et les variables aléatoires t_i, $1 \leq i \leq M$, sont les instants où se produisent un événement. Les s_i, $1 \leq i \leq M$, sont les délais ou les temps d'attente entre deux événement successifs. On dit que $(t_i)_{1 \leq i \leq M}$ définit un processus ponctuel. Désignons par X_t le nombre d'événement qui se sont produits au cours de la période de temps $[0, t]$ et supposons que $X(0) = X_0 = 0$. On définit la fonction aléatoire de comptage X^T=$\{X_t, 1 \leq t \leq T\}$ du processus ponctuel $\{t_i, 1 \leq i \leq M\}$ de la façon suivante:

$$X_t = \sum_{j=1}^{M} \mathbb{I}_{\{t_j \leq t\}},$$

X_t est ainsi le nombre d'événements qui se sont produits avant l'instant t

- Notons que X_0=0 puisque $t_1 > 0$, $X_T < \infty$. S'il n'y pas de point (d'événement) dans l'intervalle [0,T], alors on pose M=0 et X_t=0, $1 \leq t \leq T$. Bien entendu, M=X_T.

- Pour $0 \leq s \leq t$, $X_t - X_s$ est le nombre d'événements qui se sont produits dans l'intervalle de temps $]s, t]$.

- La trajectoire X^T est continue à droite et admet une limite à gauche. C'est une fonction croissante, constante par morceaux avec des sauts de hauteur 1 c'est-à-dire $X_t = X_{t-}$+(1 ou 0).

-Notons que la donnée $\{X_t, 1 \leq t \leq T\}$ est équivalente à celle de la suite $\{t_i, 1 \leq i \leq M\}$, et que pour tout entier n, l'on a les relations suivantes:

1) $\{X_t \geq n\} = \{t_n \leq t\}$

2) $\{X_t = n\} = \{t_n \leq t \leq t_{n+1}\}$

3) $\{X_s < n \leq X_t\} = \{s < t_n \leq t\}$.

Processus de Poisson homogène

On dit que le processus ponctuel $\{t_i, 1 \leq i \leq M\}$ ou sa fonction aléatoire comptage X^T est un processus de Poisson homogène si X^T est une fonction aléatoire à accroissements indépendants et stationnaires c'est-à-dire si

a) X_0=0 p.s.;

b) quels que soient $0 \leq s_0 < s_1 < \cdots < s_N \leq T$, les accroissements $X_{s_2} - X_{s_1}, \cdots, X_{s_N} - X_{s_{N-1}}$ du processus sur des intervalles disjoints $[s_1, s_2]$,$\cdots$, $[s_N, s_{N-1}]$ sont des variables aléatoires indépendantes;

c) pour $0 \leq s < t$, $X_t - X_s \sim \mathfrak{L}(Poisson)$ et cette loi ne dépend de s et de t que par la différence $t - s$

La propriété b) est appelée la stationnarité des accroissements de $\{X_t\}$. La définition du processus de Poisson est justifiée par la proposition:

Soit $X^T = \{X_t, 0 \leq t \leq T\}$ la fonction aléatoire de comptage d'un processus de Poisson homogène. Il existe $\lambda > 0$ tel que pour tous $0 \leq s < t$, la loi de $X_t - X_s$ est la loi de Poisson de paramètre $\lambda(t-s)$, c'est-à-dire

$$\mathbf{P}(X_t - X_s = k) = e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k}, \quad k \in \mathbb{N}$$

Nous remarquons également que ce paramètre λ est appelé l'intensité du processus de Poisson homogène $\{X_t, 0 \leq t \leq T\}$. Il est égal au nombre moyen d'événements qui se produisnet pendant un intervalle de temps de longueur unité, ce qui signifie:

$$\mathbb{E}(X_{t+1} - X_t) = \lambda.$$

Processus de Poisson non homogène

Soit $(\Omega, \mathcal{F}, \mathbf{P})$ un espace de probabilité. Un processus stochastique $X^T = \{X_t, 0 \leq t \leq T\}$ défini sur cet espace est un processus de Poisson non homogène de mesure d'intensité $\Lambda(\cdot, \cdot)$ si les conditions suivantes sont satisfaites:

- X_0=0 p.s.
- Pour tout $i \in \mathbb{N}$ et pour tout $0 \leq t_0 < t_1 < \ldots < t_i = T$, les variables aléatoires $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_i} - X_{t_{i-1}}$ sont indépendantes.
- Pour tout $0 \leq s < t \leq T$ il existe une fonction $\Lambda(s,t) \geq 0$ telle que pour tout $k \in \mathbb{N}$ on ait:

$$\mathbf{P}(X_t - X_s = k) = e^{-\Lambda(t,s)} \frac{\Lambda^k(t,s)}{k}.$$

On remarque également que

1. la mesure d'intensité $\Lambda(\cdot, \cdot)$ est absolument continue, si elle est de la forme

$$\Lambda(t,s) = \int_s^t \lambda(u) du,$$

où $\lambda(u)$, $u \geq 0$ est une fonction non négative. La fonction $\lambda(\cdot)$ est alors appelée fonction d'intensité.

2. Le processus de Poisson $\{X_t, 0 \leq t \leq T\}$ de mesure d'intensité $\Lambda(\cdot, \cdot)$ vérifie

$$\mathbb{E}(X_t) = \Lambda(t) \quad et \quad Var(X_t) = \Lambda(t)$$

où on a posé $\Lambda(t)$= $\Lambda(0,t)$.

3. Si $\lambda(u)$ est une constante c'est-à-dire $\lambda(u) = \lambda > 0$, on a un processus de Poisson homogène

$$\mathbf{P}(X_t - X_s = k) = e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k}, \quad k \in \mathbb{N}$$

Processus de Poisson dans le cadre général:

Soit $(\Omega, \mathcal{F}, \mathbf{P})$ un espace de probabilité et $(\mathfrak{X}, \rho)$ un espace métrique complet (ρ une métrique) muni de la tribu des boréliens. Notons par $\mathcal{M}$ l'espace des mesures σ-finies définies sur $(\mathfrak{X}, \rho)$ et par $\mathcal{M}_0$ le sous-espace des mesures σ-finies définies sur $(\mathfrak{X}, \rho)$ et prenant ses valeurs dans $\mathbb{N}$, c'est-à-dire que:

$$X \in \mathcal{M}_0 \Longleftrightarrow X = \sum_i \delta_{t_i}$$

où $t_i \in \mathfrak{X}$ et δ_t est la mesure de Dirac au point t.

Notons par $\mathbb{B}(\mathfrak{X})$ la plus σ-algèbre des sous-ensembles de $\mathcal{M}$ telle que:

$$\Pi_B : \mathcal{M}_0 \to \mathbb{N},$$

avec $\Pi_B(X) = X(B)$, $B \in \mathbb{B}$ soit mesurable. Soit $\Lambda \in \mathcal{M}$. Une variable aléatoire X définie sur $(\Omega, \mathcal{F}, \mathbf{P})$ et à valeurs dans $\mathcal{M}_0$ est un processus de Poisson de mesure d'intensité Λ si et seulement si on a:

- Pour chaque choix des ensembles finis disjoints $B_1, B_2, \cdots, B_m \in \mathbb{B}$, les variables aléatoires $X(B_1), X(B_2), \cdots, X(B_m)$ sont indépendantes.
- Pour tout $B \in \mathbb{B}$ avec $\Lambda(B) < \infty$, X(B) est une variable aléatoire qui suit la loi de Poisson de paramètre $\Lambda(B)$ c'est-à-dire que

$$\mathbf{P}(X(B) = k) = e^{-\Lambda(B)} \frac{\Lambda(B)^k}{k}, \quad k \in \mathbb{N}$$

Exemples.

Voici quelques exemples d'applications de processus de Poisson non homogènes. Pour une présentation exhaustive des processus de Poisson non homogènes voir Kutoyants [41].

La désintégration radioactive. L'émission de photons par une source radioactive peut être considérée comme un processus de Poisson de fonction d'intensité

$$\lambda(t) = ae^{-bt}, \qquad t \geq 0,$$

où $a > 0$ dépend de la quantité et du type de la source et $b > 0$ est la moyenne de survie de la source.

La théorie de la fiabilité. La suite d'échecs

$$\lambda(t) = at^b, \qquad t \geq 0,$$

où $b > 1$, correspond au cas où les échecs deviennent de plus en plus fréquents. Un processus de Poisson avec une telle intensité est appelé processus de Weibull.

La détection optique. le flux de photons produit lorsque un rayon de lumière est concentré sur une surface photosensible peut être modélisé par un processus de Poisson non homogène (voir Mandel [45]). Il existe trois cas particuliers intéressants pour les télécommunications optiques et les systèmes de radar:

- *Modélisation d'amplitude.* La fonction d'intensité du processus de Poisson est définie par

$$\lambda(\vartheta, t) = \vartheta g(t) + \lambda_0, \qquad t \geq 0$$

où $g(\cdot)$ est une fonction positive connue et le paramètre $\lambda > 0$ supposé connu est appelé courant d'obscurité.

- *Modélisation de phase, télémétrie optique.* Le processus de Poisson décrivant la vitesse de génération des électrons à la sortie d'un photon détecteur est défini par

$$\lambda(\vartheta, t) = g(t - \vartheta) + \lambda_0, \qquad t \geq 0$$

où $g(\cdot)$ et λ_0 sont définis comme précédemment.

- *Modélisation de fréquence, vitesse de télémétrie optique.* Dans une démarche visant à mesurer la vitesse d'un objet, l'intensité d'un faisceau de lumière dirigée vers celui-ci est modulée sinusoïdalement. La lumière réfléchie a toute sa fréquence décalée en raison de l'effet du Doppler, la fréquence de modulation est décalée par une quantité proportionnelle à la modulation de fréquence et à la mesure des variations des taux de la lumière réfléchie de l'objet. La vitesse de génération des électrons à la sortie d'un photo-détecteur utilisé pour observer la lumière réfléchie est alors de la forme

$$\lambda(\vartheta, t) = \alpha\{1 + m\cos[2\pi(\omega_m + \vartheta)t]\} + \lambda_0, \qquad t \geq 0,$$

où α et m sont des constantes $(\alpha > 0, \mid m \mid < 1)$, ω_m est la modulation de fréquence, et ϑ est l'effet de Doppler.

La diversité des modèles suppose aussi la diversité d'études de problèmes statistiques d'identification de ces modèles.

1.3 Intégrale stochastique et rapport de vraisemblance

Intégrale stochastique

Soient $(\Omega, \mathcal{F}, \mathbf{P})$ un espace probabilisé et $X^T = \{X(t),\ 0 \le t \le T\}$ un processus de Poisson de mesure d'intensité $\Lambda(\cdot)$ sur $\mathbb{R}_+$.

Si la mesure d'intensité du processus de Poisson $\Lambda(\cdot)$ est absolument continue, alors il existe une fonction $\lambda(\cdot)$ telle que

$$\Lambda([a,b]) = \int_a^b \lambda(t)\, dt.$$

La fonction $\lambda(\cdot)$ est appelée fonction d'intensité du processus de Poisson. Cette fonction est positive ou nulle.

Notons $L_p(\Lambda)$, $p \ge 1$, l'ensemble des fonctions mesurables $f : [0,T] \to \mathbb{R}$ telles que

$$\int_0^T |f(t)|^p\, \Lambda(dt) < +\infty.$$

Soit $f \in L_1(\Lambda)$, on définie

$$I(f) = \int_0^T f(t)\, dX(t)$$

comme l'intégrale de Lebesgue–Stieltjes car la fonction $\{X(t),\ 0 \le t \le T\}$ est croissante et à variation finie. Si $t_1, \cdots, t_m$ sont les instants du processus de Poisson alors

$$I(f) = \sum_{0 \le t_i \le T} f(t_i).$$

On peut définir l'intégrale stochastique par rapport au processus de Poisson centré $\Pi(t) = X(t) - \Lambda(t)$ sur l'intervalle $[0,T]$ par

$$I_\star(f) = \int_0^T f(t)\, d\Pi(t) = I(f) - \int_0^T f(t)\Lambda(dt).$$

Énonçons quelques propriétés de ces intégrales qui joueront un rôle important dans la suite.

Proposition 1 *Soit $f(\cdot) \in L_1(\Lambda)$, alors les intégrales stochastiques*

$$I(f) = \int_0^T f(t)\, dX(t), \qquad I_\star(f) = \int_0^T f(t)\, d\Pi(t)$$

sont bien définies et vérifient

$$\mathbf{E} I(f) = \int_0^T f(t)\, \Lambda(dt), \qquad \mathbf{E} I_\star(f) = 0.$$

Les fonctions caractéristiques sont

$$\phi(\mu) = \mathbf{E} \exp\left\{i\mu I(f)\right\} = \exp\left\{\int_0^T \left[\exp\left\{i\mu f(t)\right\} - 1\right] \Lambda(dt)\right\}$$

et

$$\phi_\star(\mu) = \mathbf{E} \exp\left\{i\mu I_\star(f)\right\} = \exp\left\{\int_0^T \left[\exp\left\{i\mu f(t)\right\} - 1 - i\mu f(t)\right] \Lambda(dt)\right\}$$

Si les fonctions $f(\cdot), g(\cdot) \in L_1(\Lambda) \bigcap L_2(\Lambda)$ alors

$$\mathbf{E} I_\star(f)^2 = \int_0^T f(t)^2\, \Lambda(dt), \qquad \mathbf{E}\big(I_\star(f)\, I_\star(g)\big) = \int_0^T f(t) g(t)\, \Lambda(dt).$$

Pour toute fonction $f(\cdot) \in L_1(\Lambda)$ telle que $e^{f(\cdot)} - 1 - f(\cdot) \in L_1(\Lambda)$, nous avons

$$\mathbf{E} \exp\left\{\int_0^T f(t)\, \Pi(dt)\right\} = \exp\left\{\int_0^T \left[e^{f(t)} - 1 - f(t)\right] \Lambda(dt)\right\}$$

Preuve : voir Kutoyants [41]□

Le polynôme des moments de $I_\star(f)$ peut être majoré à l'aide de la proposition suivante

Proposition 2 *Soit $f(\cdot) \in L_{2p}(\Lambda)$, alors il existe une constante $C_p > 0$ indépendante de $f(\cdot)$ et de $\Lambda(\cdot)$ telle que :*

$$\mathbf{E} \left(\int_0^T f(t)\, \Pi(dt)\right)^{2p} \leq C_p \left\{\int_0^T f(t)^{2p}\, \Lambda(dt) + \left(\int_0^T f(t)^2\, \Lambda(dt)\right)^p\right\}$$

Preuve : Voir Kutoyants [41, Lemme 1.2, page 21]□

Sur la filtration naturelle $(\mathbb{F}_t)_{t\in[0,T]}$ définie par $\mathbb{F}_t = \sigma\left\{X_s \,|s \leq t\right\}$, les processus $\Pi(t) = X(t) - \Lambda(t)$ et $M(t) = X(t)^2 - \Lambda(t)$ possèdent les propriétés suivantes:

Proposition 3

1. *$\Pi(t)$ est une $\mathbb{F}_t$–martingale réelle de carré intégrable, continue à droite et possédant une limite à gauche.*
2. *$M(t)$ est une $\mathbb{F}_t$–martingale.*
3. *Si $f(\cdot) \in L_1(\Lambda)$, le processus*

$$\eta_t = \int_0^t f(u)\Pi(du) = \int_0^T 1_{\{u<t\}}\, f(u)\Pi(du)$$

est une $\mathcal{F}_t$–martingale.

4. *Si $f(\cdot)$ est une fonction bornée, alors pour tout $N > 0$, nous avons:*

$$\mathbf{P}\left(\sup_{0\le t\le T} \eta_t > N\right) \le \exp\left\{-N + \frac{1}{2}\int_0^T \left|e^{f(t)} - 1 - f(t)\right| \Lambda(dt)\right\}$$

et

$$\mathbf{P}\left(\sup_{0\le t\le T} |\eta_t| > N\right) \le \exp\left\{-N + \frac{1}{2}\int_0^T \left|e^{f(t)} - 1 - f(t)\right| \Lambda(dt)\right\} \; +$$

$$+ \;\exp\left\{-N + \frac{1}{2}\int_0^T \left|e^{-f(t)} - 1 + f(t)\right| \Lambda(dt)\right\}.$$

Rapport de vraisemblance

Considérons $(D[0,T], \mathcal{D}_T)$ l'espace mesurable des fonctions continues à droite, admettant une limite à gauche, ayant au plus des discontinuités de première espèce, définies sur l'intervalle $[0,T]$ et munie de sa tribu borélienne $\mathcal{D}_T$.

Soient X_1 et X_2 deux processus de Poisson de mesure d'intensité

$$\Lambda^{(1)} = \left\{\Lambda^{(1)}(t),\; 0 \le t \le T\right\}$$

et

$$\Lambda^{(2)} = \left\{\Lambda^{(2)}(t),\; 0 \le t \le T\right\}$$

satisfaisant les conditions suivantes

$$\int_0^T \Lambda^{(1)}(t)dt < \infty, \qquad \int_0^T \Lambda^{(2)}(t)dt < \infty.$$

Soient $\Lambda_1(\cdot)$ et $\Lambda_2(\cdot)$ les deux mesures d'intensités correspondantes sur l'espace $([0,T], \mathcal{B}_T)$ où $\mathcal{B}_T$ est la σ-algèbre de Borel. Les deux processus de Poisson ainsi définis apparetiennent à $(D[0,T], \mathcal{D}_T)$ et induits des mesures de probabilités notées respectivement $\mathbf{P}_1$ et $\mathbf{P}_2$. Notons respectivement $\mathbf{E}_1$ et $\mathbf{E}_2$ les espérances mathématiques par rapport à ces probabilités.

Notons $\mathbf{P}_1 \perp \mathbf{P}_2$ la singularité des mesures, $\mathbf{P}_1 \ll \mathbf{P}_2$ l'absolue continuité des mesures et $\mathbf{P}_1 \sim \mathbf{P}_2$ l'équivalence des mesures.

Si $\Lambda_1 \ll \Lambda_2$, alors il existe une dérivée de Radon-Nykodim $\lambda(t) = \frac{d\Lambda_1}{d\Lambda_2}(t)$.

Proposition 4 *Si* $\Lambda_1 \ll \Lambda_2$ *et* $\Lambda_2([0,T]) < +\infty$ *alors* $\mathbf{P}_1 \ll \mathbf{P}_2$.

De plus

$$\frac{d\mathbf{P}_1}{d\mathbf{P}_2}(X) = \exp\left\{\int_0^T \ln \lambda(t)\, X(dt) - \int_0^T [\lambda(t) - 1]\, \Lambda_2(dt)\right\}.$$

Enfin si $\Lambda_1 \sim \Lambda_2$ *alors* $\mathbf{P}_1 \sim \mathbf{P}_2$.

Preuve : Voir Brown [6] □

Soient X, X_1 et X_2 trois processus de Poisson d'intensité respectives Λ_0, Λ_1 et Λ_2 et définissant les mesures de probabilité respectives $\mathbf{P}_0$, $\mathbf{P}_1$ et $\mathbf{P}_2$. Si les mesures Λ_1 et Λ_2 sont équivalentes à la mesure Λ_0 i.e. $\Lambda_1 \ll \Lambda$ et $\Lambda_2 \ll \Lambda$, alors la proposition précédente implique l'existence des rapports de vraisemblance

$$Z_1 = \frac{d\mathbf{P}_1}{d\mathbf{P}_0}(X)$$

et

$$Z_2 = \frac{d\mathbf{P}_2}{d\mathbf{P}_0}(X).$$

Les mesures Λ_i, $i = 1, 2$ sont définies pour tout borélien $\mathcal{B}$ dans $\mathbb{R}$ de sorte que

$$\Lambda_i(\mathcal{B}) = \int_{\mathcal{B}} \lambda_i(u)\, \Lambda(du)$$

où $\lambda_i(\cdot)$ est la fonction d'intensité associée à Λ_i.

Nous avons alors la proposition suivante

Proposition 5 *Si* $\Lambda_1 \sim \Lambda_2$ *sur l'intervalle* $[0,T]$*, alors*

$$\mathbf{E}_0 \left| Z_1^{\frac{1}{2}} - Z_2^{\frac{1}{2}} \right| \leq \int_0^T \left(\sqrt{\lambda_1(t)} - \sqrt{\lambda_2(t)}\right)^2 \Lambda(dt)$$

et

$$\mathbf{E}_0 Z_1^{\frac{1}{2}} \leq \exp\left\{-\frac{1}{2}\int_0^T \left(\sqrt{\lambda_1(t)-1}\right)^2 \Lambda(dt)\right\}$$

où $\mathbf{E}_0(\cdot)$ *est l'espérance mathématique par rapport à la mesure de probabilité* $\mathbf{P}_0$.

De plus pour tout $p>1$*, nous avons*

$$\mathbf{E}_0\left|Z_1^{\frac{1}{2p}} - Z_2^{\frac{1}{2p}}\right|^p \leq a_p\left\{\left(\int_0^T l^2(t)\,\Lambda_1(dt)\right)^p + \left(\int_0^T l^2(t)\,\Lambda_2(dt)\right)^p + \right.$$

$$+ \int_0^T l^{2p}(t)\,\Lambda_1(dt) + \int_0^T l^{2p}(t)\,\Lambda_2(dt)\Big\} +$$

$$+ \ (2p)^{-2p}\left\{\int_0^T l^2(t)\,\Lambda_1(dt) + \int_0^T l^2(t)\,\Lambda_2(dt)\right\},$$

où $a_p = \frac{1}{2}p^{-2p}C_p$ *et la fonction* $l(t) = \ln\left(\lambda_2(t)\,\lambda_1(t)^{-1}\right)$.

Preuve : Grâce à la proposition 4, la première inégalité s'obtient en écrivant

$$\mathbf{E}_0\left|Z_1^{\frac{1}{2}} - Z_2^{\frac{1}{2}}\right|^2 = \mathbf{E}_1\left|(Z_2 Z_1^{-1})^{\frac{1}{2}} - 1\right|^2 = 2 - 2\mathbf{E}_1(Z_2 Z_1^{-1})^{\frac{1}{2}} =$$

$$= 2 - 2\exp\left\{\frac{1}{2}\int_0^T (\lambda_1(t)-\lambda_2(t))\ \Lambda(dt)\right\}\mathbb{E}_1\exp\left\{\int_0^T \ln\left(\frac{\lambda_2(t)}{\lambda_1(t)}\right)^{\frac{1}{2}} X(dt)\right\}$$

$$= 2 - 2\exp\left\{\frac{1}{2}\int_0^T \left(2\sqrt{\lambda_1(t)\lambda_2(t)} - \lambda_1(t) - \lambda_2(t)\right)\ \Lambda(dt)\right\}$$

$$= 2 - 2\exp\left\{-\frac{1}{2}\int_0^T \left(\sqrt{\lambda_1(t)} - \sqrt{\lambda_2(t)}\right)^2 \Lambda(dt)\right\},$$

où $\mathbf{E}_1(\cdot)$ et $\mathbf{E}_2(\cdot)$ représentent respectivement les espérances mathématiques par rapport aux mesures de probabilités $\mathbf{P}_1$ et $\mathbf{P}_2$. Compte tenu de l'inégalité $1-e^{-x} \leq x$ pour tout $x \geq 0$, il s'en suit

$$\mathbf{E}\left|Z_1^{\frac{1}{2}} - Z_2^{\frac{1}{2}}\right|^2 \leq \int_0^T [\sqrt{\lambda_1(t)} - \sqrt{\lambda_2(t)}]^2\,\Lambda(dt),$$

La deuxième inégalité découle de la première en posant $\Lambda_2 = \Lambda$. Pour établir la dernière inégalité, nous utilisons l'inégalité suivante valable pour tout p et tout x positifs :

$$\left(x^{\frac{1}{2p}} - 1\right)^{2p} \leq (2p)^{-2p} (\ln x)^{2p} (1 + x).$$

En effet, nous avons

$$\begin{aligned}
\mathbf{E} \left| Z_1^{\frac{1}{2p}} - Z_2^{\frac{1}{2p}} \right|^{2p} &= \mathbf{E}_1 \left| (Z_2 Z_1^{-1})^{\frac{1}{2p}} - 1 \right|^{2p} \leq \\
&\leq (2p)^{-2p} \mathbf{E}_1 \left\{ \left(\ln(Z_2 Z_1^{-1}) \right)^{2p} (1 + Z_2 Z_1^{-1}) \right\} = \\
&= (2p)^{-2p} \mathbf{E}_1 \left(\ln(Z_2 Z_1^{-1}) \right)^{2p} + (2p)^{-2p} \mathbf{E}_2 \left(\ln(Z_2 Z_2^{-1}) \right)^{2p}
\end{aligned}$$

En utilisant l'inégalité

$$x - 1 - \ln\, x \leq \frac{1}{2} (\ln\, x)^2 (1 + x)$$

et la proposition 2, la dernière inégalité peut être estimée de la façon suivante

$$\begin{aligned}
\mathbf{E}_2 \left(\ln(Z_2 Z_1^{-1}) \right)^{2p} = \mathbf{E}_2 \Bigg\{ \int_0^T \ln \left(\frac{\lambda_1(t)}{\lambda_2(t)} \right) \, M(dt) \, - \\
- \int_0^T \left[\lambda_1(t) - \lambda_2(t) - \lambda_2(t) \ln \frac{\lambda_1(t)}{\lambda_2(t)} \right] \, \Lambda(dt) \Bigg\}^{2p} \leq \\
\leq 2^{2p-1} \mathbf{E}_2 \left\{ \int_0^T \ln \left(\frac{\lambda_1(t)}{\lambda_2(t)} \right) \, M(dt) \right\}^{2p} + \\
+ 2^{2p-1} \left\{ \int_0^T \left[\lambda_1(t) - \lambda_2(t) - \lambda_2(t) \ln \frac{\lambda_1(t)}{\lambda_2(t)} \right] \, \Lambda(dt) \right\}^{2p} \leq \\
\leq 2^{2p-1} C_p \left\{ \int_0^T l^{2p}(t) \, \Lambda_2(dt) + \left(\int_0^T l^2(t) \, \Lambda_2(dt) \right)^p \right\} + \\
+ \frac{1}{2} \left\{ \int_0^T l^2(t) \, \Lambda_2(dt) + \int_0^T l^2(t) \, \Lambda_1(dt) \right\}^{2p} \square
\end{aligned}$$

1.4 Estimation paramétrique: Cas régulier et non régulier

L'inférence statistique en théorie asymptotique repose sur deux branches fondamentales. La première est la théorie de l'estimation et la seconde est la théorie des tests. La théorie de l'estimation est caractrisée par trois approches: l'approche paramétrique, semi paramétrique ou non paramétrique.

Estimation paramétrique pour les modèles réguliers

En estimation paramétrique, les études sont orientées en fonction de la régularité du modèle. Autrement dit, dans les situations où la fonction d'intensité présente des points de discontinuités(sauts) ou non. En situation régulière, dans le cas i.i.d ([32]) comme dans le cas des processus stochastiques (voir [40], [41]), les familles de mesure sont localement asymptotiquement normales. De ce fait, il a été démontré que les estimateurs tels que celui du maximum de vraisemblance, Bayesien et de la distance minimale sont consistants, asymptotiquement normaux et sous certaines conditions, asymptotiquement efficaces. Par exemple dans le i.i.d., si on considère $\varphi(\epsilon)$, ($\varphi(\epsilon) > 0$, $|\varphi(\epsilon)| \to 0$ lorsque $\epsilon \to 0$) une certaine matrice de normalisation, $\hat{u}$ et $\widetilde{u}$ vérifiant

$$Z(\hat{u}) = \max_u Z(u) \qquad \widetilde{u} = \frac{\int_{\mathbb{R}^k} l(u)Z(u)du}{\int_{\mathbb{R}^k} Z(u)du}$$

$l(\cdot)$ étant une certaine fonction de perte et $Z(\cdot)$ le processus limite du modèle. Ainsi, sous certaines conditions les estimateurs Bayésien $\widetilde{\theta}_\epsilon$ et du maximum de vraismeblance $\hat{\theta}_\epsilon$ vérifient les relations suivantes: la convergence en distribution

$$\varphi^{-1}(\epsilon)(\hat{\theta}_\epsilon - \theta) \Rightarrow \hat{u}, \quad \varphi^{-1}(\epsilon)(\widetilde{\theta}_\epsilon - \theta) \Rightarrow \widetilde{u}$$

et la convergence des moments

$$\lim_{\epsilon\to 0} \mathbf{E}_\theta^{(\epsilon)} w\left(\varphi^{-1}(\epsilon)(\hat{\theta}_\epsilon - \theta)\right) \Rightarrow \mathbf{E}w\left(\hat{u}\right), \quad \lim_{\epsilon\to 0} \mathbf{E}_\theta^{(\epsilon)} w\left(\varphi^{-1}(\epsilon)(\widetilde{\theta}_\epsilon - \theta)\right) \Rightarrow \mathbf{E}w\left(\widetilde{u}\right);$$

ceci uniformément dans $\theta \in \mathbf{K}$ ($\mathbf{K}$ un compact de l'ensemble des paramètres Θ et $w(\cdot)$ une fonction de perte).

En théorie asymptotique, plusieurs résultats ont été aussi obtenus. On peut citer par exemple un modèle issu des télécommunications optiques. Le flux de photons produit quand un rayon de lumière est concentré sur une surface photosensible peut être modélisé par un processus de Poisson non homogène (voir Mandel [45]), avec une fonction d'intensité de la forme:

$$\lambda(\theta, t) = \theta g(t) + \lambda_0, \qquad t \geq 0,$$

où $g(\cdot)$ est une fonction positive connue et le paramètre $\lambda > 0$ supposé connu est appelé courant d'obscurité. Hoversten et Snyder [31], Bar David [3] et Kutoyants [39] ont étudié le problème d'estimation de l'amplitude θ. On obtient pour ces types de modèles des résultats assez proches de la statistique classique des variables aléatoires. Sous certaines conditions, l'estimateur du maximum de vraisemblance (EMV), l'estimateur Bayésien (EB) et celui de la distance minimale (EDM) du paramètre θ du modèle sont consistants, asymptotiquement normaux et leurs moments d'ordre p convergent. C'est-à-dire pour l'estimateur θ_n

$$\mathbf{P}_\theta - \lim_{n\to+\infty} \theta_n = \theta,$$

$$\mathcal{L}_\theta\left\{ I^{\frac{1}{2}}(\theta)\,(\theta_n - \theta) \right\} \Rightarrow \mathcal{L}\{\zeta\}$$

$$\lim_{n\to+\infty} \mathbf{E}|I^{\frac{1}{2}}(\theta)\,(\theta_n - \theta)\,|^p = \mathbf{E}|\zeta|^p,$$

où $I(\theta)$ est l'information de Fisher du modèle

$$I(\theta) = \int_A \frac{\partial\lambda(\theta,t)}{\partial\theta}\frac{\partial\lambda(\theta,t)^T}{\partial\theta}\lambda(\theta,t)^{-1}dt$$

A est la fenêtre d'observation du processus, J est une matrice unité, et $\mathcal{L}\{\zeta\} = \mathbb{N}(0, J)$.

Estimation paramétrique pour les modèles de ruptures

La situation est différente dans le cas i.i.d. comme dans le cas d'un processus stochastique car le modèle est caractérisé par une fonction d'intensité discontinue et la famille de mesure correspondante n'est pas localement asymptotiquement normale. Dans ce cas les propriétés du MLE et du BE sont différentes de celles décrites dans le cas régulier. En plus le MLE n'est pas asymptotiquement efficace. Par ailleurs, des études sont faites sur les modèles de ruptures pour les processus de Poisson avec un saut de taille fixe. Kutoyants, dans ([41] a considéré un modèle où l'intensité modulée $S(t)$, $t \geq 0$ a une discontinuité (un saut) à un certain point τ_0 de la période et le récepteur détecte les photons correspondant au processus de Poisson d'une certaine intensité. Ainsi, il montre que cette forme de modulation de phase est essentiellement meilleure que la transmission avec la modulation de fréquence avec une fonction périodique régulière (voir Exemple 2.2 dans [41]). Les vitesses de convergences dans les problèmes de modulation de phase et de fréquence avec une intensité discontinue sont plus grandes que les vitesses dans les cas réguliers. C'est-à-dire les exemple 2.2 et 2.3 dans [41]. Dans la même optique, Dabye dans [10] a considéré plusieurs problèmes d'estimation paramétriques bidimensionnelle pour des modèles particuliers d'un processus de Poisson non homogène. Pour ces modèles, le rapport de vraisemblance limite est un log-processus de Poisson. Nous notons également des modèles de rupture pour les processus de diffusion. Ainsi pour le modèle de signal dans un bruit blanc gaussien, le rapport de vraisemblance limite est un log-processus de Wienner. L'estimation des positions de singularités (singularités de type cusp, type "0" et type "∞") a été considéré dans plusieurs cas d'étude notamment dans le cas i.i.d. (voir Ibragimov et Khasminskii [32]) mais aussi dans le cas des processus stochastiques avec Dachian dans ([14],[16] [15] et [19]), en collaboration avec Kutoyants ou avec Negri pour la plupart(dénégéresecnce, explosions, explosions de la dérive).

1.5 Tests d'ajustements

Le problème des tests d'ajustements est l'un des thèmes centraux en pratique et en théorie statistique. Il est important de vérifier le degré de correspondance entre les résultats observés et les résultats attendus car étant le fondement de la statistique classique. L'approche non paramétrique classique liée ces problèmes de tests d'hypothèses peut être trouvée dans Durbin [24] , Greenwood and Nikulin [30], Lehmann et Romano [44]. Les tests les plus connus sont: le test de Kolmogorov-Smirnov, le test Cramér-von Mises et le test du Chi-deux. L'avantage de ces tests classiques s'explique par leur caractère "distribution-free", c'est-à-dire la distribution limite de la statistique en question ne dépend pas du modèle de base choisie. Cette propriété nous permet d'obtenir un seuil universel qui peut être utilisé pour n'importe quel modèle. Dans [34], Insgter et Kutoyants ont étudié un test d'hypothèse non paramétrique pour une intensité d'un processus de Poisson. Leur travail est une extension de celui de Insgter [33] aux processus de Poisson. En effet, Insgter et Suslina [35] avaient fait les tests identiques mais pour un modèle de bruit blanc gaussien. Dans le même champs Dachian et Kutoyants dans [18] avaient présenté plusieurs résultats concernant les tests de Kolmogorov-Smirnov et de Cramér-von Mises pour quelques processus à temps continue. Comme modèles, ils considèrent une équation différentielle avec un petit bruit, un processus de diffusion ergodique, un processus de Poisson "self-exciting" et un processus Poisson. Pour chaque modèle ils proposent un test qui fournit de niveau asymptotique $\alpha \in (0,1)$ et décrivent le comportement asymptotique de la fonction de puissance sous des alternatives locales. Par exemple:

Supposons qu'on observe n processus de Poisson indépendants $X^{(n)} = (X_1, \ldots, X_n)$, où $X_j = (X_j(t), t \in \mathbb{R})$ sont les trajectoires d'un processus de Poisson avec la fonction moyenne $\Lambda(t) = \mathbb{E}X_j(t)$. Rappelons que si l'hypothèse de base(nulle) est simple i.e.

$$\mathscr{H}_0 \qquad : \qquad \Lambda(\cdot) = \Lambda_0(\cdot),$$

où $\Lambda_0(\cdot)$ est une fonction connue avec $\Lambda_0(\infty) < \infty$ et l'alternative

$$\mathscr{H}_1 \qquad : \qquad \Lambda(\cdot) \neq \Lambda_0(\cdot),$$

alors la statistique de type Cramér-von Mises est définie comme suit:

$$\widetilde{\Delta}_n = \frac{n}{\Lambda_0(\infty)^2} \int_{-\infty}^{+\infty} \left[\widehat{\Lambda}_n(t) - \Lambda_0(t)\right]^2 d\Lambda_0(t).$$

Ici

$$\widehat{\Lambda}_n(t) = \frac{1}{n}\sum_{j=1}^{n} X_j(t)$$

est la moyenne empirique du processus de Poisson. Ainsi ils ont démontré entre autres que la statistique converge vers la limite suivante:

$$\widetilde{\Delta}_n \Longrightarrow \Delta \equiv \int_0^1 W(s)^2\, ds.$$

où $W(s), 0 \leq s \leq 1$ est processus de Wienner standard.

Par conséquent le test

$$\widetilde{\psi}_n(X^n) = \mathbb{1}_{\left\{\widetilde{\Delta}_n > c_\varepsilon\right\}}, \qquad \mathbb{P}\{\Delta > c_\varepsilon\} = \varepsilon,$$

est asymptotiquement "distribution-free". La valeur $\varepsilon \in (0,\, 1)$ est la taille ou niveau du test.

Cependant, en pratique, les hypothèses à tester sont parfois de nature plus complexes. Les premiers travaux orientés aux problèmes de tests d'ajustement avec des hypothèses composées en statistique classique sont dus à Rao (voir aussi Durbin [24]). Ce dernier avait proposé un test d'hypothèse composée dans le cas où la fonction de distribution dépend des paramètres inconnus multidimensionnels. Ainsi, l'hypothèse de base devient composée, c'est-à-dire qu'elle ne détermine pas la distribution de l'échantillon d'une manière unique. Lorsque les paramètres sont estimés, les tests de Kolmogorov-Smirnov et de Cramér-von Mises ne sont pas "distribution-free". Il s'en suit que les valeurs critiques changent d'une hypothèse à une autre. Différentes valeurs du paramètre entrainent différentes valeurs critiques souvent au sein d'une même famille paramétrique. La propriété de "distribution-free" devient alors cruciale puisque les valeurs critiques sont calculées une seule fois pour n'importe quelle distribution définie sous l'hypothèse. Pour surmonter cette difficulté plusieurs méthodes ont été développées. Rao (voir aussi Durbin [24]) a suggéré la méthode de l'échantillon divisé. Le problème de Durbin entraine une transformation martingale du processus empirique qui fût proposé par Khamaladge [36]. L'approche martingale de Khamaladge [36] permet de construire des tests d'hypothèses asymptotiquement "distribution- free". Cette approche est utilisée par différents auteurs dont Bai [4] dans les modèles de régression, Koenker et Xiao [37]. D'autres méthodes (méthodes de translation) ont été développées pour étudier le caractère asymptotiquement "parameter-free" c'est-à-dire la distribution limite de la statistique de test sous l'hypothèse de base ne dépend pas du parmètre inconnu.

Chapter 2

On Multiple change-point estimation for Poisson process: case of non zero jumps sum

2.1 Introduction

This work is devoted to the problem of parameter estimation by the observations of n independent of inhomogeneous Poisson processes. It is supposed that these processes have the same intensity function and this intensity function has two points of discontinuity(jumps). The positions of these jumps depends on the unknown one-dimensional parameter and we have to estimate the value of this parameter. Our goal is to describe the asymptotic behavior of the maximum likelihood estimator(MLE) and the Bayesian estimator (BE). We show that the rate of convergence of these estimators is n and the limit distributions are different. We propose a lower bound on the mean-square risk of all estimators and then we show that the BE is asymptotically efficient in the sense of this bound. We realize the program of the study of such estimators developed by Ibragimov and Khaminski [32]. Therefore we show that the normalized likelihood ratio process converges to some random process, which is exponential functional of four Poisson processes with constant intensities. The statistical estimation problems with discontinuous intensity function are in some sense close to the similar problem of parameter estimation by independent observations of the random variables with discontinuous density function. Such study was initiated in the work of Chernov and Rubin [8](discontinuous density with one jump). The case of many discontinuities was studied in the work of Rubin [52], see as well Ermakov [27]. Further development can be found in the works of Ibragimov and Khaminski [32], Strasser [53] and Pflug [50]. For the Poisson process with discontinuous intensity function the disorder-type hypothesis testing problem was studied in [28]. The problem of parameter estimation (consistency, limit distributions, convergence of moments, asymptotic efficiency) was considered in [40] and [41]. Note as well the related statistical problems in the works [25],[1],[2],[12],[15] and the references therein.

This work is a continuation of the study initiated in [40]. The considered model of observation with intensity function $\lambda(\theta_0, t) = \lambda_0 + \lambda_1(t)\mathbb{1}_{\{\theta_0 \leq t \leq \theta_0+\tau_0\}}$ is typical for statistical radiophysics. Here $\lambda_1(t)\mathbb{1}_{\{\theta_0 \leq t \leq \theta_0+\tau_0\}}$ is an signal of length $\tau_0 > 0$ and $\lambda_0 > 0$ is some Poissonian noise. Therefore the problem of estimation of the parameter θ corresponds to the evaluation of the moment of arriving of the signal. We suppose that the sum of the jumps $\lambda_1(\theta_0) - \lambda_1(\theta_0 + \tau_0) \neq 0$. The case $\lambda_1(\theta_0) - \lambda_1(\theta_0 + \tau_0) = 0$ will be considered in Chapter 3.

Note that the general theory of parameter estimation for inhomogeneous Poisson processes is now well developed, see, e.g.; [40],[41] and the reference therein. In the regular situation (the intensity function is smooth with respect to the parameter) it is shown that these estimators are consistent, asymptotically normal and asymptotically efficient. The corresponding family of measure is locally asymptotically normal (LAN), i.e., the normalized likelihood ratio $Z_n(u)$ weakly converges to the process $Z(u) = \exp\{u\Delta - u^2/2\}$. However the situation is different when the intensity function has discontinuities. In this this case the limit $Z(u)$ of the likelihood ratios $Z_n(u)$ contain the Poisson processes, like $Z(u) = \exp\{\ln \rho x_+(u) - ru\}$, where $x_+(u)$ is a Poisson process. This difference in limits describes the difference in limit distributions of the MLE and BE. Recall that the limit of the likelihood ratio process in the case of change-point estimation for signals observed in the white Gaussian noise is $Z(u) = \exp\{W(u) - |u|/2\}$, $W(u)$ is a Wiener process [32].

For our model, the normalised likelihood ratio converges to the difference of two log Poisson type processes and we use it to show that the Bayesian and maximum likelihood estimators are consistent, converge in law and their moments converge also.

The intensity function of the observed Poisson process has two jumps and their position depends on the unknown parameter θ. Note that such model can be used in optical communication theory: the parameter (information) θ is a transmitted through the Poissonian channel with modulated intensity where λ_0 is the intensity of the noise [40]. In section 2.2 we describe the details of the model of observation and give the definitions of estimators.The section 2.3 is devoted to the main results (the properties of estimators and the proofs). Section 2. 4 consists of simulations.

2.2 Statement of the problem and some preliminary results

We suppose that the observations $X^{(n)} = (X_1, \ldots, X_n)$ are n independent inhomogeneous Poisson processes $X_j = \left\{X_j(t),\, 0 \le t \le T\right\}$, $j = 1, \ldots, n$ with the same intensity function

$$\lambda\left(\theta, t\right) = \lambda_0 + \lambda_1(t)1\!\!1_{\{\theta \le t \le \theta + \tau_0\}}, \quad 0 \le t \le \tau.$$

Here

$$\theta \in \Theta = (\alpha, \beta), \quad \tau = T - \tau_0, \quad 0 < \alpha < \beta < \beta + \tau_0 < \tau.$$

Under this condition we have two jumps of the intensity function on the interval of observations. Recall that

$$\mathbf{E}_\theta X_j(t) = \Lambda\left(\theta, t\right) = \int_0^t \lambda\left(\theta, s\right)\, ds.$$

The parameter θ is supposed to be unknown and we have to estimate it by the observations $X^{(n)}$. We are interested by the asymptotic ($n \to \infty$) behavior of the MLE and the BE.

Denote by $\mathbf{P}_\theta^{(n)}$ the measure induced in the space of observations by n realizations of the Poisson process with the intensity function $\lambda\left(\theta, t\right)$, $0 \le t \le \tau$. As $\lambda_0 > 0$ and $\lambda_1(t)$ is bounded, the measures $\mathbf{P}_\theta^{(n)}$, $\theta \in \Theta$ are equivalent and the likelihood ratio function

$$L\left(\theta, \theta_1, X^{(n)}\right) = \frac{d\mathbf{P}_\theta^{(n)}}{d\mathbf{P}_{\theta_1}^{(n)}}(X^{(n)}) \quad \theta \in \Theta$$

is

$$\begin{aligned} L\left(\theta, \theta_1, X^{(n)}\right) \;=\; & \exp\Bigg\{\sum_{j=1}^n \int_0^\tau \ln\Big(\frac{\lambda_0 + \lambda_1(t)1\!\!1_{\{\theta \le t \le \theta + \tau_0\}}}{\lambda_0 + \lambda_1(t)1\!\!1_{\{\theta_1 \le t \le \theta_1 + \tau_0\}}}\Big) dX_j(t) \\ & -n \int_0^\tau \left(\lambda_1(t)1\!\!1_{\{\theta \le t \le \theta + \tau_0\}} - \lambda_1(t)1\!\!1_{\{\theta_1 \le t \le \theta_1 + \tau_0\}}\right) dt\Bigg\}. \end{aligned}$$

Here $\theta_1 \in \Theta$ is some fixed value.

To show the shape of such likelihood ratio function we can take the result of the simulations given below in the Section 2.5, where $\lambda\left(\theta, t\right) = 1 + 2t1\!\!1_{\{\theta \le t \le \theta + 2\}}$.

A realization of such log likelihood ratio in the case $n = 1$ and $\theta_0 = 2$ is given on the figure below.

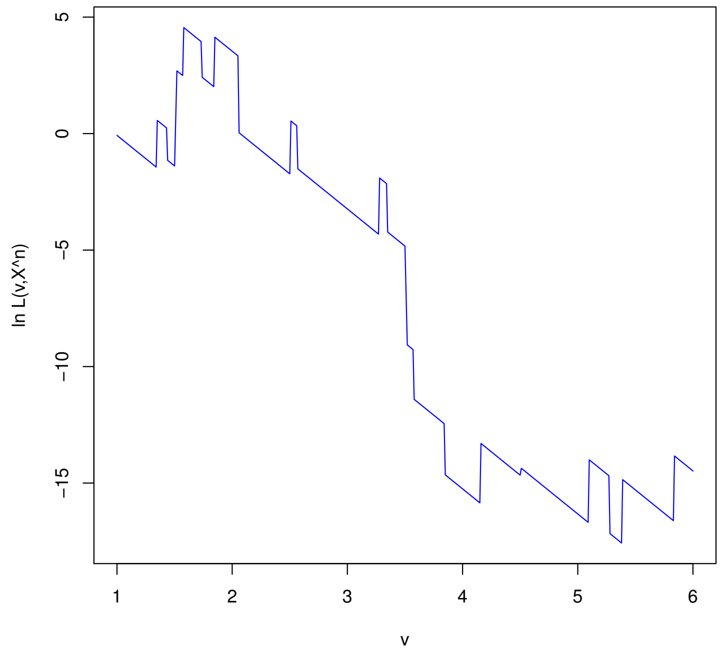

Figure 2.1: A realization of $\ln L\left(\theta, X^{(n)}\right)$ with $n = 1$

.

As the likelihood ratio $L\left(\theta, \theta_1, X^{(n)}\right)$ is a discontinuous function of θ we define the MLE $\hat{\theta}_n$ as a solution of the following equation

$$\max\left\{L\left(\hat{\theta}_n+, \theta_1, X^{(n)}\right), L\left(\hat{\theta}_n-, \theta_1, X^{(n)}\right)\right\} = \sup_{\theta\in\Theta} L\left(\theta, \theta_1, X^{(n)}\right).$$

Here $L\left(\hat{\theta}_n+, \theta_1, X^{(n)}\right)$ and $L\left(\hat{\theta}_n-, \theta_1, X^{(n)}\right)$ are the left and the right limits of the function $L\left(\theta, \theta_1, X^{(n)}\right)$ at the point $\hat{\theta}_n$ respectively.

To introduce the Bayesian estimator we suppose that the unknown parameter is a random variable with known, positive and continuous density function $p(\theta)$, $\theta \in \Theta$. Then BE $\widetilde{\theta}_n$ is a conditional expectation, which can be written as follows

$$\widetilde{\theta}_n = \mathbf{E}\left(\theta / X^{(n)}\right) = \int_\alpha^\beta \theta p(\theta) L\left(\theta, X^{(n)}\right)\, d\theta \left(\int_\alpha^\beta p(\theta) L\left(\theta, X^{(n)}\right)\, d\theta\right)^{-1}.$$

To describe the properties of estimators, we need additional notations. Let

$$Z_{\theta_0}(u)=\begin{cases}\exp\Big\{\rho_1(\theta_0)\;X^+(u)+\rho_2(\theta_0)\;Y^+(u)-r(\theta_0)u\Big\}, & u\geq 0,\\ \exp\Big\{-\rho_1(\theta_0)\;X^-(-u)-\rho_2(\theta_0)\;Y^-(-u)-r(\theta_0)u\Big\}, & u<0\end{cases}$$

where $X^+(\cdot)$, $X^-(\cdot)$, $Y^+(\cdot)$ and $Y^-(\cdot)$ are independent Poisson processes on $\mathbb{R}_+$ of the constant intensities $\lambda_0+\lambda_1(\theta_0)$, λ_0, λ_0 and $\lambda_0+\lambda_1(\theta_0+\tau_0)$ respectively. The parameters $\rho_1(\theta_0)$, $\rho_1(\theta_0)$ and $r(\theta_0)$ are defined as follows

$$\rho_1(\theta_0)=\ln\frac{\lambda_0}{\lambda_0+\lambda_1(\theta_0)},\quad \rho_2(\theta_0)=\ln\frac{\lambda_0+\lambda_1(\theta_0+\tau_0)}{\lambda_0},\quad r(\theta_0)=\lambda_1(\theta_0+\tau_0)-\lambda_1(\theta_0).$$

Denote $\rho_1=\rho_1(\theta_0)$, $\rho_1=\rho_1(\theta_0)$, $r=r(\theta_0)$. Now let us note, that up to a linear change time, the process $Z_{\theta_0}(\cdot)$ is nothing but the process $Z^*_\rho(\cdot)$ with $\rho=(\rho_1,\rho_2)$. Indeed, by putting $u=\frac{v}{r}$, $X_1^\pm(v)=X^\pm(\frac{v}{r})$ and $Y_1^\pm(v)=Y^\pm(\frac{v}{r})$ we get

$$Z^*_\rho(v):=\begin{cases}\exp\Big\{\rho_1X_1^+(v)+\rho_2Y_1^+(v)-v\Big\}, & v\geq 0,\\ \exp\Big\{-\rho_1X_1^-(-v)-\rho_2Y_1^-(-v)-v\Big\}, & v<0\end{cases}$$

where $X_1^+(\cdot)$, $X_1^-(\cdot)$, $Y_1^+(\cdot)$ and $Y_1^-(\cdot)$ are independent Poisson processes on $\mathbb{R}_+$ of intensities $\frac{\lambda_0e^{-\rho_1}}{r}$, $\frac{\lambda_0}{r}$, $\frac{\lambda_0}{r}$ and $\frac{\lambda_0e^{\rho_2}}{r}$ respectively.

Introduce the random variables $\hat{u}$, $\hat{u}_\rho$, $\widetilde{u}$ and $\widetilde{u}_\rho$ by the equations

$$\max\{Z_{\theta_0}(\hat{u}-),Z_{\theta_0}(\hat{u}+)\}=\sup_{u\in\mathbb{R}}Z_{\theta_0}(u),$$

$$\max\left\{Z^*_\rho(\hat{u}_\rho-),Z^*_\rho(\hat{u}_\rho+)\right\}=\sup_{v\in\mathbb{R}}Z^*_\rho(v),$$

$$\widetilde{u}=\int_{-\infty}^{+\infty}uZ_{\theta_0}(u)\,du\left(\int_{-\infty}^{+\infty}Z_{\theta_0}(u)\,du\right)^{-1}$$

and

$$\widetilde{u}_\rho=\int_{-\infty}^{+\infty}vZ^*_\rho(v)\,dv\left(\int_{-\infty}^{+\infty}Z^*_\rho(v)\,dv\right)^{-1}.$$

Let us note that $\hat{u}\equiv\frac{\hat{u}_\rho}{r}$ and $\widetilde{u}\equiv\frac{\widetilde{u}_\rho}{r}$.

2.3 Asymptotic properties of Bayesian estimator

2.3.1 Mains results

Introduce the condition C_0:

- *The constants* λ_0 *and* τ_0 *are strictly positive and known.*
- *The function* $\lambda_1(\cdot)$, $t \in [0, \tau]$ *is strictly increasing, strictly positive and continuous.*

The condition of increasing $\lambda_1(\cdot)$ is introduced to avoid the situation when the sum of two jumps $\lambda_1(\theta_0) - \lambda_1(\theta_0 + \tau_0) = 0$ for all $\theta_0 \in \Theta$. It is sufficient to require that

$$\inf_{\theta \in \Theta} |\lambda_1(\theta + \tau_0) - \lambda_1(\theta)| > 0.$$

The first result gives us the lower bound on the risk of all the estimators.

Theorem 1 *Let the condition* C_0 *be fulfilled. Then for all* $\theta_0 \in \Theta$

$$\lim_{\delta \to 0} \underline{\lim}_{n \to +\infty} \inf_{\overline{\theta}_n} \sup_{|\theta - \theta_0| < \delta} n^2 \mathbf{E}_\theta \left(\overline{\theta}_n - \theta\right)^2 \geq \mathbf{E}_{\theta_0} \widetilde{u}^2 = \frac{\mathbf{E}_{\theta_0}\left(\widetilde{u}_\rho^2\right)}{r^2}. \tag{2.1}$$

Here the inf is taken over all possible estimators $\overline{\theta}_n$ of the parameter θ.

The inequality (2.1) allows us to give the following definition.

Let the condition C_0 *be satisfied, we say that the estimator* $\overline{\theta}_n$ *is asymptotically efficient, if for all* $\theta_0 \in \Theta$ *we have*

$$\lim_{\delta \to 0} \lim_{n \to +\infty} \sup_{|\theta - \theta_0| < \delta} n^2 \mathbf{E}_\theta \left(\overline{\theta}_n - \theta\right)^2 = \frac{\mathbf{E}_{\theta_0}\left(\widetilde{u}_\rho^2\right)}{r^2}.$$

Suppose that the unknown parameter θ is a random variable with a continuous and positive *prior* density $p(\theta)$, $\theta \in \Theta$. Denote $\mathbf{K} \subset \Theta$ a compact set. Then the Bayesian estimator $\widetilde{\theta}_n$ has the following properties.

Theorem 2 *Let the condition* C_0 *be fulfilled. Then the Bayesian estimator* $\widetilde{\theta}_n$ *verify uniformly on* $\theta_0 \in \mathbf{K}$ *the relations:*

it is consistent

$$\mathbf{P}_{\theta_0} - \lim_{n \to +\infty} \widetilde{\theta}_n = \theta_0,$$

it converges in law

$$\mathcal{L}_\theta\left\{n\left(\widetilde{\theta}_n-\theta_0\right)\right\}\Rightarrow\mathcal{L}\left(\frac{\widetilde{u}_\rho}{r}\right).$$

For any $p>0$ the moments of $\widetilde{\theta}_n$ converge

$$\lim_{n\rightarrow+\infty}\mathbf{E}_{\theta_0}|n\left(\widetilde{\theta}_n-\theta_0\right)|^p=\mathbf{E}_{\theta_0}\frac{|\widetilde{u}_\rho|^p}{|r|^p}$$

and this estimator is asymptotically efficient.

The uniform consistency is understood as follows: for any $\nu>0$

$$\lim_{n\rightarrow\infty}\sup_{\theta_0\in\mathbf{K}}\mathbf{P}_{\theta_0}^{(n)}\left(|\widetilde{\theta}_n-\theta_0|>\nu\right)=0.$$

2.3.2 Proofs of theorems

The presented proofs are based on the general results of Ibragimov and Khasminski in [32] and follow the application of their results to a model of inhomogeneous Poisson process given in ([41], Chapter 5).

Introduce the normalized likelihood ratio

$$\begin{aligned}
Z_{\theta_0,n}(u) &\equiv L\left(\theta_0+\frac{u}{n},\theta_0,X^{(n)}\right)\\
&= \exp\left\{\sum_{j=1}^n\int_0^\tau\ln\left(\frac{\lambda_0+\lambda_1(t)\mathbb{1}_{\left\{\theta_0+\frac{u}{n}\leq t\leq\theta_0+\frac{u}{n}+\tau_0\right\}}}{\lambda_0+\lambda_1(t)\mathbb{1}_{\{\theta_0\leq t\leq\theta_0+\tau_0\}}}\right)dX_j(t)\right.\\
&\left.\quad -n\int_0^\tau\left(\lambda_1(t)\mathbb{1}_{\left\{\theta_0+\frac{u}{n}\leq t\leq\theta_0+\frac{u}{n}+\tau_0\right\}}-\lambda_1(t)\mathbb{1}_{\{\theta_0\leq t\leq\theta_0+\tau_0\}}\right)dt\right\}
\end{aligned}$$

where $u\in U_n=\left(n\left(\alpha-\theta_0\right),n\left(\beta-\theta_0\right)\right)$.

To prove the Theorems 1 and 2 we need the following lemmas.

Lemma 1 *Let the condition C_0 be satisfied, then the finite dimensional distributions of the process $Z_{\theta_0,n}(u)$ converge to the finite dimensional distributions of the process $Z_{\theta_0}(u)$ and this convergence is uniform with respect to $\theta_0\in\mathbf{K}$.*

Proof. Suppose that $u>0$ (the other case can be treated in a similar way) and put $u=u_*$ (with fixed $u_*>0$). Denote Δ and Δ_{u_*} the intervals defined by $\Delta=\{\theta_0\leq t\leq\theta_0+\tau_0\}$ and $\Delta_{u_*}=\left\{\theta_0+\frac{u_*}{n}\leq t\leq\theta_0+\tau_0+\frac{u_*}{n}\right\}$ respectively.

Then we can write

$$\begin{aligned}\ln Z_{\theta_0,n}(u_*) &= \sum_{j=1}^{n}\int_0^\tau \ln\Big(\frac{\lambda_0+\lambda_1(t)\mathbb{1}_{\{t\in\Delta_{u_*}\}}}{\lambda_0+\lambda_1(t)\mathbb{1}_{\{t\in\Delta\}}}\Big)dX_j(t) - \\ &\quad -n\int_0^\tau \left[\lambda_1(t)\mathbb{1}_{t\in\Delta(u_*)}-\lambda_1(t)\mathbb{1}_{\{t\in\Delta\}}\right]dt \equiv A_n - B_n,\end{aligned}$$

with obvious notations.

The characteristic function of $\ln Z_{\theta_0,n}(u_*)$ is calculated as follows

$$\Phi_n(y) = \mathbf{E}_{\theta_0}\exp\left(iy\ln Z_{\theta_0,n}(u_*)\right) = \exp\left(-iyB_n\right)\mathbf{E}_{\theta_0}\exp\left(iyA_n\right).$$

We remark that A_n is a sum of independents random variables, that is,

$$A_n = \sum_{j=1}^{n}\int_0^\tau \ln\left(\frac{\lambda_0+\lambda_1(t)\mathbb{1}_{\{t\in\Delta_{u_*}\}}}{\lambda_0+\lambda_1(t)\mathbb{1}_{\{t\in\Delta\}}}\right)dX_j(t),$$

where the processes X_j are independents.

Therefore

$$\begin{aligned}&\mathbf{E}_{\theta_0}\exp\left(iyA_n\right)\\ &\quad=\exp\left\{n\int_0^\tau\left[\exp\left(iy\ln\frac{\lambda_0+\lambda_1(t)1_{\{t\in\Delta_{u_*}\}}}{\lambda_0+\lambda_1(t)1_{\{t\in\Delta\}}}\right)-1\right]\left(\lambda_0+\lambda_1(t)1_{\{t\in\Delta\}}\right)dt\right\}.\end{aligned}$$

The value u_* is fixed, therefore $\frac{u_*}{n}\to 0$ as $n\to+\infty$. Thus we have $\tau_0>\frac{u_*}{n}$. The interval $[0,\tau]$ can be expressed as the sum of five intervals $[0,\theta_0]$, $[\theta_0,\theta_0+\frac{u_*}{n}]$, $[\theta_0+\frac{u_*}{n},\theta_0+\tau_0]$, $[\theta_0+\tau_0,\theta_0+\frac{u_*}{n}+\tau_0]$ and $[\theta_0+\frac{u_*}{n}+\tau_0,\tau]$.

The calculation of $\ln\mathbf{E}_{\theta_0}\exp\{iyA_n\}$ gives

$$\begin{aligned}\ln\mathbf{E}_{\theta_0}e^{iyA_n} &= n\int_{\theta_0}^{\theta_0+\frac{u_*}{n}}\left[\exp\left(iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1(t)}\right)-1\right]\left(\lambda_0+\lambda_1(t)\right)dt\\ &\quad+n\int_{\theta_0+\tau_0}^{\theta_0+\frac{u_*}{n}+\tau_0}\left[\exp\left(iy\ln\frac{\lambda_0+\lambda_1(t)}{\lambda_0}\right)-1\right]\lambda_0 dt.\end{aligned}$$

Further

$$\begin{aligned}B_n &= n\int_0^\tau\left(\lambda_1(t)\mathbb{1}_{\{t\in\Delta_{u_*}\}}-\lambda_1(t)\mathbb{1}_{\{t\in\Delta\}}\right)dt\\ &= -n\int_{\theta_0}^{\theta_0+\frac{u_*}{n}}\lambda_1(t)dt+n\int_{\theta_0+\tau_0}^{\theta+\frac{u_*}{n}+\tau_0}\lambda_1(t)dt.\end{aligned}$$

Using the mean value theorem for the integrals, it is possible to find some $\widetilde{\theta}_{1,n} \in \left(\theta_0, \theta_0 + \frac{u_*}{n}\right)$ and $\widetilde{\theta}_{2,n} \in \left(\theta_0 + \tau_0, \theta_0 + \frac{u_*}{n} + \tau_0\right)$ such that

$$\begin{aligned}\ln \mathbf{E}_{\theta_0} e^{iyA_n} &= u_* \left[\exp\left(iy \ln \frac{\lambda_0}{\lambda_0 + \lambda_1(\widetilde{\theta}_{1,n})}\right) - 1\right] \left(\lambda_0 + \lambda_1(\widetilde{\theta}_{1,n})\right) \\ &+ u_* \left[\exp\left(iy \ln \frac{\lambda_0 + \lambda_1(\widetilde{\theta}_{2,n})}{\lambda_0}\right) - 1\right]\end{aligned}$$

and

$$B_n = -u_* \lambda_1(\widetilde{\theta}_{1,n}) + u_* \lambda_1(\widetilde{\theta}_{2,n}).$$

The function $\lambda_1(\cdot)$ is continuous. Therefore, when n tend to infinity, $\widetilde{\theta}_{1,n}$ and $\widetilde{\theta}_{2,n}$ converge to θ_0 and $\theta_0 + \tau_0$ respectively. Thus we obtain the following limits

$$B_n \longrightarrow -u_* \lambda_1(\theta_0) + u_* \lambda_1(\theta_0 + \tau_0) = r u_* \tag{2.2}$$

and

$$\begin{aligned}\ln \mathbf{E}_{\theta_0} e^{iyA_n} \to\ & u_* \left[\exp\left(iy \ln \frac{\lambda_0}{\lambda_0 + \lambda_1(\theta_0)}\right) - 1\right] (\lambda_0 + \lambda_1(\theta_0)) \\ & + u_* \left[\exp\left(iy \ln \frac{\lambda_0 + \lambda_1(\theta_0 + \tau_0)}{\lambda_0}\right) - 1\right] \lambda_0.\end{aligned} \tag{2.3}$$

Combining the relations (2.3) and (2.2), we obtain

$$\begin{aligned}\Phi_n(y) \longrightarrow \Phi(y) = \exp\Bigg\{ & u_* \left[\exp\left(iy \ln \frac{\lambda_0}{\lambda_0 + \lambda_1(\theta_0)}\right) - 1\right] (\lambda_0 + \lambda_1(\theta_0)) \\ & + \left[u_* \exp\left(iy \ln \frac{\lambda_0 + \lambda_1(\theta_0 + \tau_0)}{\lambda_0}\right) - 1\right] \lambda_0 - r u_* \Bigg\}.\end{aligned} \tag{2.4}$$

To end this proof we will verify that for $u_* > 0$ the characteristic function of $Z_{\theta_0}(u_*)$ coincides $\Phi(y)$.

Indeed, for $u_* > 0$ we have

$$\begin{aligned}\mathbf{E}_{\theta_0} e^{iy \ln Z_{\theta_0}(u_*)} = \mathbf{E}_{\theta_0} \exp\Bigg\{ & iy \ln \left(\frac{\lambda_0}{\lambda_0 + \lambda_1(\theta_0)}\right) X^+(u_*) \\ & + iy \ln \left(\frac{\lambda_0 + \lambda_1(\theta_0 + \tau_0)}{\lambda_0}\right) Y^+(u_*) - \mathrm{i} y u_* r \Bigg\}\end{aligned}$$

By the independency of the Poisson processes $X^+(\cdot)$ and $Y^+(\cdot)$, we have the following representation

$$\mathbf{E}_{\theta_0} e^{iy\ln Z_{\theta_0}(u_*)} = \mathbf{E}_{\theta_0} \exp\left\{ iy\ln\left(\frac{\lambda_0}{\lambda_0+\lambda_1(\theta_0)}\right) X^+(u_*)\right\} \times \mathbf{E}_{\theta_0} \exp\left\{ iy\ln\left(\frac{\lambda_0+\lambda_1(\theta_0+\tau_0)}{\lambda_0}\right) Y^+(u_*)\right\} \times \exp\left\{-\mathrm{i}yu_*r\right\}$$

which allows us to calculate the characteristic function of each Poisson process $X^+(\cdot)$ and $Y^+(\cdot)$. Therefore

$$\begin{aligned}
&\mathbf{E}_{\theta_0} \exp\left\{ iy\ln\left(\frac{\lambda_0}{\lambda_0+\lambda_1(\theta_0)}\right) X^+(u_*)\right\} \\
&\quad = \exp\left\{ \int_0^{u_*} \left[\exp\left(iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1(\theta_0)}\right) - 1\right] (\lambda_0+\lambda_1(\theta_0))\, dt\right\} \\
&\quad = \exp\left\{ u_* \left[\exp\left(iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1(\theta_0)}\right) - 1\right] (\lambda_0+\lambda_1(\theta_0))\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}_{\theta_0} \exp\left\{ iy\ln\left(\frac{\lambda_0+\lambda_1(\theta_0+\tau_0)}{\lambda_0}\right) Y^+(u_*)\right\} \\
&\quad = \exp\left\{ \int_0^{u_*} \left[\exp\left(iy\ln\frac{\lambda_0+\lambda_1(\theta_0+\tau_0)}{\lambda_0}\right) - 1\right] \lambda_0 dt\right\} \\
&\quad = \exp\left\{ u_* \left[\exp\left(iy\ln\frac{\lambda_0+\lambda_1(\theta_0+\tau_0)}{\lambda_0}\right) - 1\right] \lambda_0\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\exp\left\{ u_* \left[\exp\left(iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1(\theta_0)}\right) - 1\right] (\lambda_0+\lambda_1(\theta_0))\right\} \\
&\qquad \times \exp\left\{ u_* \left[\exp\left(iy\ln\frac{\lambda_0+\lambda_1(\theta_0+\tau_0)}{\lambda_0}\right) - 1\right] \lambda_0\right\} \exp\left\{ iyu_*r\right\} \\
&\qquad\quad = \exp\left\{ u_* \left[\exp\left(iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1(\theta_0)}\right) - 1\right] (\lambda_0+\lambda_1(\theta_0)) \right. \\
&\qquad\qquad \left. + u_* \left[\exp\left(iy\ln\frac{\lambda_0+\lambda_1(\theta_0+\tau_0)}{\lambda_0}\right) - 1\right] \lambda_0 - \mathrm{i}yu_*r\right\};
\end{aligned}$$

which is equal to the relation (2.4).

For $u<0$, by a similar way we show that

$$\begin{aligned}\mathbf{E}_{\theta_0}\exp\{iy\ln Z_{\theta_0,n}(u)\}\to\exp\bigg\{&-u\left[\exp\left(iy\ln\frac{\lambda_0+\lambda_1(\theta)}{\lambda_0}\right)-1\right]\lambda_0\\&-u\left[\exp\left(iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1(\theta_0+\tau_0)}\right)-1\right](\lambda_0+\lambda_1(\theta_0+\tau_0))-\mathrm{i}yur\bigg\}\\&=\mathbf{E}_{\theta_0}\exp\bigg\{iy\ln\frac{\lambda_0+\lambda_1(\theta_0)}{\lambda_0}X^-(-u)+iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1(\theta_0+\tau_0)}Y^-(-u)\\&\quad-\mathrm{i}yur\bigg\}=\mathbf{E}_{\theta_0}\exp\bigg\{iy\ln Z_{\theta_0}(u)\bigg\}.\end{aligned}$$

The convergence of the multi dimensional distribution of the vector $(Z_{\theta_0,n}(u_1), Z_{\theta_0,n}(u_2)\ldots,Z_{\theta_0,n}(u_k))$ to the vector $(Z_{\theta_0}(u_1),Z_{\theta_0}(u_2),\ldots,Z_{\theta_0}(u_k))$ for $k\geq 2$ can be shown as follows. According to the Wold device, we verify the convergence of the sum $\sum_{l=1}^K a_l\ln Z_{\theta_0,n}(u_l)$ for any vector $a=(a_1,\cdots,a_K)$ to the sum $\sum_{l=1}^K a_l\ln Z_{\theta_0}(u_l)$. Indeed, we have to show that the characteristic function

$$\Phi_n(y_1,\cdots,_K)=\mathbf{E}_{\theta_0}\exp\bigg\{iy_1a_1\ln Z_{\theta_0,n}(u_1)+\cdots,+iy_Ka_K\ln Z_{\theta_0,n}(u_K)\bigg\}$$

converges to the characteristic function of the limit vector

$$\Phi(y_1,\cdots,_K)=\mathbf{E}_{\theta_0}\exp\bigg\{iy_1a_1\ln Z_{\theta_0}(u_1)+,\cdots,+iy_Ka_K\ln Z_{\theta_0}(u_K)\bigg\}.$$

Lemma 2 *Let the condition C_0 be satisfied; then there exists a constant $C>0$ such that*

$$\sup_{\theta_0\in\mathbf{K}}\mathbf{E}_{\theta_0}\mid Z_{\theta_0,n}^{1/2}(u_1)-Z_{\theta_0,n}^{1/2}(u_2)\mid^2\leq C\mid u_1-u_2\mid;$$

for all $n\in\mathbb{N}$, $u_1,u_2\in U_n$.

Proof. Suppose that $0\leq u_1\leq u_2$. First we consider the case $\frac{u_i}{n}<\tau_0$ for $i=1,2$.

Thus, according to the proposition 5 (Chapter 1), we have

$$\begin{aligned}
&\mathbf{E}_{\theta_0} \mid Z^{1/2}_{\theta_0,n}(u_1) - Z^{1/2}_{\theta_0,n}(u_2) \mid^2 \\
&\qquad \le n\int_0^\tau \left[\sqrt{\frac{\lambda_0+\lambda_1(t)1_{\{t\in\Delta_{u_1}\}}}{\lambda_0+\lambda_1(t)1_{\{t\in\Delta\}}}} - \sqrt{\frac{\lambda_0+\lambda_1(t)1_{\{t\in\Delta_{u_2}}}{\lambda_0+\lambda_1(t)1_{\{t\in\Delta\}}}}\right]^2 \lambda(\theta_0,t)dt \\
&\qquad \le n\int_0^\tau \frac{\left(\sqrt{\lambda_0+\lambda_1(t)1_{\{t\in\Delta_{u_1}\}}} - \sqrt{\lambda_0+\lambda_1(t)1_{\{t\in\Delta_{u_2}\}}}\right)^2}{\left(\sqrt{\lambda_0+\lambda_1(t)1_{\{t\in\Delta\}}}\right)^2}\lambda(\theta_0,t)dt \\
&\qquad \le n\int_0^\tau \left(\sqrt{\lambda_0+\lambda_1(t)1_{\{t\in\Delta_{u_1}\}}} - \sqrt{\lambda_0+\lambda_1(t)1_{\{t\in\Delta_{u_2}\}}}\right)^2 dt \\
&\qquad = n\int_{\theta_0+\frac{u_1}{n}}^{\theta_0+\frac{u_2}{n}} \left(\sqrt{\lambda_0+\lambda_1(t)} - \sqrt{\lambda_0}\right)^2 dt \\
&\qquad\quad + n\int_{\theta_0+\frac{u_1}{n}+\tau_0}^{\theta_0+\frac{u_2}{n}+\tau_0} \left(\sqrt{\lambda_0+\lambda_1(t)} - \sqrt{\lambda_0}\right)^2 dt.
\end{aligned}$$

The function $\lambda_1(\cdot)$ is strictly positive and bounded (because strictly positive and continuous on $[0,\tau]$). Therefore there exists a constant C_1 such that

$$n\int_{\theta_0+\frac{u_1}{n}}^{\theta_0+\frac{u_2}{n}} \left(\sqrt{\lambda_0+\lambda_1(t)} - \sqrt{\lambda_0}\right)^2 dt \le C_1 \mid u_2 - u_1 \mid$$

and

$$n\int_{\theta_0+\frac{u_1}{n}+\tau_0}^{\theta_0+\frac{u_2}{n}+\tau_0} \left(\sqrt{\lambda_0+\lambda_1(t)} - \sqrt{\lambda_0}\right)^2 dt \le C_1 \mid u_2 - u_1 \mid .$$

Hence

$$\mathbf{E}_{\theta_0} \mid Z^{1/2}_{\theta_0,n}(u_1) - Z^{1/2}_{\theta_0,n}(u_2) \mid^2 \le C \mid u_2 - u_1 \mid .$$

Consider also the case $\tau_0 \le \frac{u_1}{n}$ and $\tau_0 + \frac{u_1}{n} < \frac{u_2}{n}$.

$$\begin{aligned}
&n\int_0^\tau \left(\sqrt{\lambda_0+\lambda_1(t)\mathbb{1}_{\{t\in\Delta_{u_1}\}}} - \sqrt{\lambda_0+\lambda_1(t)\mathbb{1}_{\{t\in\Delta_{u_2}\}}}\right)^2 dt \\
&\qquad = n\int_{\theta_0+\frac{u_1}{n}}^{\theta_0+\frac{u_1}{n}+\tau_0} \left(\sqrt{\lambda_0+\lambda_1(t)} - \sqrt{\lambda_0}\right)^2 dt \\
&\qquad\quad + \; n\int_{\theta_0+\frac{u_2}{n}}^{\theta_0+\frac{u_2}{n}+\tau_0} \left(\sqrt{\lambda_0+\lambda_1(t)} - \sqrt{\lambda_0}\right)^2 dt \le C\tau_0.
\end{aligned}$$

Since $\frac{u_1}{n} + \tau_0 \leq \frac{u_2}{n}$, we have $\tau_0 \leq \frac{u_2 - u_1}{n}$. Hence

$$\mathbf{E}_{\theta_0} \mid Z^{1/2}_{\theta_0,n}(u_1) - Z^{1/2}_{\theta_0,n}(u_2) \mid^2 \leq n \frac{C(u_2 - u_1)}{n} = C(u_2 - u_1).$$

We obtain the similar results in the cases $\tau_0 < \frac{u_1}{n} < \frac{u_2}{n}$ and $\frac{u_1}{n} < \tau_0 < \frac{u_2}{n}$. Therefore for all $n \in \mathbb{N}$, $0 \leq u_1 \leq u_2$ and $\theta_0 \in \mathbf{K}$

$$\mathbf{E}_{\theta_0} \mid Z^{1/2}_{\theta_0,n}(u_1) - Z^{1/2}_{\theta_0,n}(u_2) \mid^2 \leq C \mid u_2 - u_1 \mid .$$

For the others cases (say $u_2 < 0 < u_1$ or $u_2 < u_1 < 0$), the proofs can be carried out in a similar way.

Lemma 3 *Let the condition C_0 be satisfied; then there exists a constant $c > 0$ such that*

$$\sup_{\theta_0 \in \mathbf{K}} \mathbf{E}_{\theta_0} Z^{1/2}_{\theta_0,n}(u) \leq e^{-c|u|};$$

for all $n \in \mathbb{N}$, $u \in U_n$.

Proof. Suppose that $u > 0$ (the case $u \leq 0$ can be treated in a similar way). According to the proposition 5 (Chapter 1), we have

$$\mathbf{E}_\theta Z^{1/2}_{\theta,n}(u) = \exp\left\{ \frac{-n}{2} \int_0^\tau \left(\sqrt{\frac{\lambda_0 + \lambda_1(t) 1_{\{t \in \Delta_u\}}}{\lambda_0 + \lambda_1(t) 1_{\{t \in \Delta\}}}} - 1 \right)^2 \left(\lambda_0 + \lambda_1(t) 1_{\{t \in \Delta\}}\right) dt \right\}.$$

If $\frac{u}{n} < \tau_0$, then

$$\begin{aligned}
&\int_0^\tau \left(\sqrt{\frac{\lambda_0 + \lambda_1(t) 1\!\!1_{\{t \in \Delta_u\}}}{\lambda_0 + \lambda_1(t) 1\!\!1_{\{t \in \Delta\}}}} - 1 \right)^2 \left(\lambda_0 + \lambda_1(t) 1\!\!1_{t \in \Delta_u}\right) dt \\
&\quad = \int_0^\tau \left(\sqrt{\lambda_0 + \lambda_1(t) 1\!\!1_{\{t \in \Delta_u\}}} - \sqrt{\lambda_0 + \lambda_1(t) 1\!\!1_{\{t \in \Delta\}}} \right)^2 dt \\
&\quad = \int_\theta^{\theta + \frac{u}{n}} \left(\sqrt{\lambda_0} - \sqrt{\lambda_0 + \lambda_1(t)} \right)^2 dt \\
&\quad\quad + \int_{\theta + \tau_0}^{\theta + \frac{u}{n} + \tau_0} \left(\sqrt{\lambda_0} - \sqrt{\lambda_0 + \lambda_1(t)} \right)^2 dt \\
&\quad = \int_{\theta_0}^{\theta_0 + \frac{u}{n}} \frac{\lambda_1^2(t)}{\left(\sqrt{\lambda_0} + \sqrt{\lambda_0 + \lambda_1(t)} \right)^2} dt \\
&\quad\quad + \int_{\theta_0 + \tau_0}^{\theta_0 + \frac{u}{n} + \tau_0} \frac{\lambda_1^2(t)}{\left(\sqrt{\lambda_0} + \sqrt{\lambda_0 + \lambda_1(t)} \right)^2} dt.
\end{aligned}$$

The function $\lambda_1(\cdot)$ is bounded and strictly positive. Therefore there exists a constant $l > 0$ such that $\lambda_1(t) \leq l$ for all $t \in [0, \tau]$.

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$\left(\sqrt{\lambda_0} + \sqrt{\lambda_0 + \lambda_1(t)}\right)^2 \leq \left(\sqrt{\lambda_0} + \sqrt{\lambda_0 + l}\right)^2$$
$$\leq 2\lambda_0 + 2\lambda_0 + 2l \leq 4\lambda_0 + 2l = L.$$

Therefore

$$\frac{\lambda_1^2(t)}{\left(\sqrt{\lambda_0} + \sqrt{\lambda_0 + \lambda_1(t)}\right)^2} \geq \frac{l^2}{L} = c > 0$$

and

$$\int_0^\tau \left(\sqrt{\frac{\lambda_0 + \lambda_1(t)\mathbb{1}_{\{t\in\Delta_u\}}}{\lambda_0 + \lambda_1(t)\mathbb{1}_{\{t\in\Delta\}}}} - 1\right)^2 \left(\lambda_0 + \lambda_1(t)\mathbb{1}_{\{t\in\Delta\}}\right) dt$$
$$\geq c\int_{\theta_0}^{\theta_0+\frac{u}{n}} dt + c\int_{\theta_0+\tau_0}^{\theta_0+\tau_0+\frac{u}{n}} dt = \frac{2uc}{n}.$$

Hence

$$\mathbf{E}_{\theta_0} Z_{\theta_0,n}^{1/2}(u) \leq e^{\frac{-n}{2}\frac{2uc}{n}} = e^{-c|u|}.$$

If $\frac{u}{n} > \tau_0$, then we have

$$\int_0^\tau \left(\sqrt{\frac{\lambda_0 + \lambda_1(t)\mathbb{1}_{\{t\in\Delta_u\}}}{\lambda_0 + \lambda_1(t)\mathbb{1}_{\{t\in\Delta\}}}} - 1\right)^2 \left(\lambda_0 + \lambda_1(t)1_{\{t\in\Delta\}}\right) dt$$
$$= \int_0^\tau \left(\sqrt{\lambda_0 + \lambda_1(t)\mathbb{1}_{\{t\in\Delta_u\}}} - \sqrt{\lambda_0 + \lambda_1(t)\mathbb{1}_{\{t\in\Delta\}}}\right)^2 dt$$
$$= \int_\theta^{\theta+\tau_0} \left(\sqrt{\lambda_0} - \sqrt{\lambda_0 + \lambda_1(t)}\right)^2 dt$$
$$+ \int_{\theta_0+\frac{u}{n}}^{\theta_0+\frac{u}{n}+\tau_0} \left(\sqrt{\lambda_0} - \sqrt{\lambda_0 + \lambda_1(t)}\right)^2 dt$$
$$= \int_{\theta_0}^{\theta_0+\tau_0} \frac{\lambda_1^2(t)}{\left(\sqrt{\lambda_0} + \sqrt{\lambda_0 + \lambda_1(t)}\right)^2} dt$$
$$+ \int_{\theta_0+\frac{u}{n}}^{\theta_0+\frac{u}{n}+\tau_0} \frac{\lambda_1^2(t)}{\left(\sqrt{\lambda_0} + \sqrt{\lambda_0 + \lambda_1(t)}\right)^2} dt \geq 2\tau_0 c_1.$$

Further $\theta = \theta_0 + \frac{u}{n}$ imply that $n = \frac{u}{\theta - \theta_0} \geq \frac{u}{\beta - \alpha}$. Therefore

$$\frac{n}{2}\int_0^\tau \left(\sqrt{\frac{\lambda_0 + \lambda_1(t)1\!\!1_{\{t\in\Delta_u\}}}{\lambda_0 + \lambda_1(t)1\!\!1_{\{t\in\Delta\}}}} - 1\right)^2 \left(\lambda_0 + \lambda_1(t)1\!\!1_{\{t\in\Delta\}}\right) dt \geq \frac{2u\tau_0 c_1}{2(\beta - \alpha)} = c\mid u\mid .$$

By the Theorem 1.10.2 in [32] and the Lemmas 1, 2, 3 we obtain the properties of the BE described in the Theorem 2.

Moreover, the uniform convergence of moments of the BE and the continuity of the limit risk allow us to cite Theorem 1.9.1 in [32] and therefore obtain the inequality (2.1) of the Theorem 1. Let us give here the proof of this bound in our case.

We have

$$\sup_{|\theta - \theta_0| < \delta} n^2 \mathbf{E}_\theta (\overline{\theta}_n - \theta)^2 \geq n^2 \int_{\theta_0 - \delta}^{\theta_0 + \delta} \mathbf{E}_\theta (\overline{\theta}_n - \theta)^2 p_\delta(\theta) d\theta.$$

Here we introduced a density function $(p_\delta(\theta), \theta_0 - \delta < \theta < \theta_0 + \delta)$. Let us denote by $\widetilde{\theta}_{\delta,n}$ the BE which corresponds to this density function. Then we have the inequality

$$\int_{\theta_0 - \delta}^{\theta_0 + \delta} \mathbf{E}_\theta (\overline{\theta}_n - \theta)^2 p_\delta(\theta) d\theta \geq \int_{\theta_0 - \delta}^{\theta_0 + \delta} \mathbf{E}_\theta (\widetilde{\theta}_{\delta,n} - \theta)^2 p_\delta(\theta) d\theta.$$

As we have a uniform convergence of moments for this BE, we obtain the limit

$$\lim_{n\to\infty} n^2 \int_{\theta_0 - \delta}^{\theta_0 + \delta} \mathbf{E}_\theta (\widetilde{\theta}_{\delta,n} - \theta)^2 p_\delta(\theta) d\theta = \int_{\theta_0 - \delta}^{\theta_0 + \delta} \frac{\mathbf{E}_\theta (\widetilde{u}_\rho^2)}{r(\theta)^2} p_\delta(\theta) d\theta.$$

Recall that $r(\theta) = \lambda_1(\theta + \tau_0) - \lambda_1(\theta)$ and $\mathbf{E}_\theta(\widetilde{u}_\rho^2)$ are continuous functions of θ. Therefore it is possible to verify that

$$\lim_{\delta\to 0} \int_{\theta_0 - \delta}^{\theta_0 + \delta} \frac{\mathbf{E}_\theta (\widetilde{u}_\rho^2)}{r(\theta)^2} p_\delta(\theta) d\theta = \frac{\mathbf{E}_{\theta_0} (\widetilde{u}_\rho^2)}{r^2}.$$

Therefore the bound (2.1) is proved.

2.4 Asymptotic properties of MLE

Recall the condition C_0:

- *The constants* λ_0 *and* τ_0 *are strictly positive and known.*
- *The function* $\lambda_1(\cdot)$, $t \in [0, \tau]$ *is strictly increasing, strictly positive and continuous.*

Theorem 3 *Let the condition* C_0 *be fulfilled, then the maximum likelihood estimator* $\hat{\theta}_n$ *verify uniformly on* $\theta_0 \in \mathbf{K}$ *the relations*

it is consistent

$$\mathbf{P}_{\theta_0} - \lim_{n \to +\infty} \hat{\theta}_n = \theta_0,$$

converges in law

$$\mathcal{L}_\theta \left\{ n \left(\hat{\theta}_n - \theta_0 \right) \right\} \Rightarrow \mathcal{L} \left(\frac{\hat{u}_\rho}{r} \right).$$

For any $p > 0$ *the moments of* $\hat{\theta}_n$ *converge*

$$\lim_{n \to +\infty} \mathbf{E}_{\theta_0} |n \left(\hat{\theta}_n - \theta_0 \right) |^p = \frac{\mathbf{E}_{\theta_0} |\hat{u}_\rho|^p}{|r|^p}$$

2.4.1 Weak Convergence in $\mathbf{D}_0(\mathbb{R})$

Introduce the space $\mathbf{D}_0(\mathbb{R})$ of functions $\varphi(u)$ without discontinuities of the second kind defined on $\mathbb{R}$ and such that $\lim_{|u| \to +\infty} \varphi(u)=0$. We assume that all the functions $\varphi(u) \in \mathbf{D}_0(\mathbb{R})$ are continuous from the right, and have limits from the left (càdlàg).

Let φ_1 and φ_2 be two functions belonging to $\mathbf{D}_0(\mathbb{R})$. The Skorohod distance between them is defined as follows

$$d(\varphi_1, \varphi_2) = \inf_\mu \left[\sup_{\mathbb{R}} |\varphi_1(u) - \varphi_2(\mu(u))| + \sup_{\mathbb{R}} |u - \mu(u)| \right],$$

where the inf is taken over all the increasing continuous one-to-one mappings $\mu : \mathbb{R} \longrightarrow \mathbb{R}$. This metric space $(\mathbf{D}_0(\mathbb{R}), d(\cdot, \cdot))$ is complete and separable. For $z \in \mathbf{D}_0(\mathbb{R})$, we put

$$\begin{aligned} \Delta_h(z) &= \sup_{u \in \mathbb{R}} \sup_{u-h \leq u' < u < u'' \leq u+h} \left[\min \left\{ \left| z(u') - z(u) \right|, \left| z(u'') - z(u) \right| \right\} \right] \\ &+ \sup_{|u| > h^{-1}} |z(u)| . \end{aligned}$$

For all $\theta \in \Theta$, suppose that we have a sequence $(z_{n,\theta})_{n \geq 1}$ of stochastic processes $z_{n,\theta} = \left\{ z_{n,\theta}(u), u \in \mathbb{R} \right\}$ and a process $z_\theta = \left\{ z_\theta(u), u \in \mathbb{R} \right\}$ such that the realizations of these processes belong to the space $\mathbf{D}_0(\mathbb{R})$. Denote $\mathbf{Q}_\theta^n$ and $\mathbf{Q}_\theta$ the distributions (which we suppose depending on a parameter $\theta \in \Theta$) induced on the measurable space $(\mathbf{D}_0(\mathbb{R}), \mathcal{B}(\mathbb{R}))$ by the processes $z_{n,\theta}$ and z_θ respectively. Here $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of the metric space $\mathbf{D}_0(\mathbb{R})$. A criterion of weak convergence in $\mathbf{D}_0(\mathbb{R})$ is given in the following lemma.

Lemma 4 *Let the following two conditions be satisfied.*

1- *The finite dimensional distributions of the process $z_{n,\theta}$ converge to the finite dimensional distributions of the process z_θ uniformly in $\theta \in \mathbf{K} \subset \Theta$.*

2- *For any $\epsilon > 0$, we have*

$$\lim_{h\to 0} \sup_{n\in\mathbb{N}} \sup_{\theta\in\mathbf{K}} \mathbf{Q}_\theta^n \left\{\Delta_h(z_{n,\theta}) > \epsilon\right\} = 0. \tag{2.5}$$

Then for all functionals $\phi(\cdot) \in \mathbf{D}_0(\mathbb{R})$ the distribution of $\phi(z_{n,\theta})$ converges to the distribution of $\phi(z_\theta)$ uniformly in $\theta \in \mathbf{K}$, that is, $z_{n,\theta}$ converges weakly uniformly to z_θ.

2.4.2 Consistency and convergence in law

This lemma allows us to show the consistency and the convergence in law of the MLE $\hat{\theta}_n$. Indeed we need the weak convergence of the likelihood ratio $Z_{n,\theta_0}(\cdot)$ to the process $Z_{\theta_0}(\cdot)$ in the space $\mathbf{D}_0(\mathbb{R})$. Suppose that we already proved this convergence.

For any set $B \in \mathcal{B}(\mathbb{R})$, we define on $\mathbf{D}_0(\mathbb{R})$ the functionals $\Phi_B(\cdot)$ and $\Psi_B(\cdot)$ by

$$\Phi_B(\varphi) = \sup_{u\in B} \varphi(u) \quad \text{and} \quad \Psi_B(\varphi) = \sup_{u\in B^c} \varphi(u)$$

respectively. Thus, the functionals $\Phi_B(\cdot)$ and $\Psi_B(\cdot)$ are continuous in the the Skorohod metric. Put $\hat{u}_n = n(\hat{\theta}_n - \theta_0)$. Thus we obtain

$$\begin{aligned} \mathbf{P}_{\theta_0}^{(n)} \left(\hat{u}_n \in B\right) &= \mathbf{P}_{\theta_0}^{(n)} \left\{\left(\Phi_B(Z_{n,\theta}) > \Psi_B(Z_{n,\theta_0})\right\}\right. \\ &\longrightarrow \mathbf{P}_{\theta_0} \left(\Phi_B(Z_{\theta_0}) > \Psi_B(Z_{\theta_0})\right) = \mathbf{P}_{\theta_0} \left(\hat{u}_{\Psi_B(Z_{\theta_0})} \in B\right). \end{aligned}$$

Hence the consistency and convergence in law of Theorem 3 are proved (for more details, see the proof of Theorem 1.10.1 in [32]).

For example, if $B = (-\infty, x)$, then we have the functionals

$$\Phi_B(\varphi) = \sup_{u<x} \varphi(u), \quad \Psi_B(\varphi) = \sup_{u>x} \varphi(u).$$

Then

$$\begin{aligned}\mathbf{P}_{\theta_0}^{(n)}\left(\hat{u}_n < x\right) &= \mathbf{P}_{\theta_0}^{(n)}\left(\Phi_B(Z_{n,\theta_0}) - \Psi_B(Z_{n,\theta_0}) > 0\right),\\ \mathbf{P}_{\theta_0}\left(\hat{u} < x\right) &= \mathbf{P}_{\theta_0}\left(\Phi_B(Z_{\theta_0}) - \Psi_B(Z_{\theta_0}) > 0\right).\end{aligned}$$

The functionals $\Phi_B(\cdot)$ and $\Psi_B(\cdot)$ are continuous in $\mathbf{D}_0(\mathbb{R})$ on those elements $\varphi \in \mathbf{D}_0(\mathbb{R})$ which are functions continuous in x. The realizations Z_{n,θ_0} and Z_{θ_0} are continuous in x with probability 1 and consequently Φ_B, Ψ_B are continuous with probability 1. In view of Theorem 5.3.1 in [32],

$$\mathbf{P}_{\theta_0}^{(n)}\left(\Phi_B(Z_{n,\theta_0}) - \Psi_B(Z_{n,\theta_0}) > 0\right) \longrightarrow \mathbf{P}_{\theta_0}\left(\Phi_B(Z_{\theta_0}) - \Psi_B(Z_{\theta_0}) > 0\right),$$

provided

$$\mathbf{P}_{\theta_0}\left(\Phi_B(Z_{\theta_0}) - \Psi_B(Z_{\theta_0}) = 0\right) = 0.$$

Since Z_{θ_0} possesses only one global maximum the last probability equals $\mathbf{P}_{\theta_0}\left(\hat{u} = x\right) = 0$ for all x but countably many x. It is proved analogously that also

$$\mathbf{P}_{\theta_0}^{(n)}\left(\hat{u}_n < x\right) \longrightarrow \mathbf{P}_{\theta_0}\left(\sup_{u<x} Z_{\theta_0}(u) > \sup_{u>x} Z_{\theta_0}(u)\right) = \mathbf{P}_{\theta_0}\left(\hat{u} < x\right).$$

The consistency is proved in the same manner.

Now our goal is to show the weak convergence. For that we just have to check conditions of Lemma 4. The convergence of the finite dimensional distributions is already checked by Lemma 1. Note that the limit process $Z_{\theta_0}(\cdot)$ as well as the likelihood ratio process $Z_{n,\theta_0}(\cdot)$ are continuous in probability.

Recall that $U_n = ((\alpha-\theta_0)\,n, (\beta-\theta_0)\,n)$, and put

$$\begin{aligned}V_n &= \left\{u \in \mathbb{R} : \theta_0 + \frac{u}{n} \in \left(\alpha - \frac{1}{n}, \beta + \frac{1}{n}\right)\right\}\\ &= \Big((\alpha-\theta_0)\,n - 1, (\beta-\theta_0)\,n + 1\Big).\end{aligned}$$

The process $Z_{n,\theta_0}(u)$ is defined on the set U_n. We extend it over the entire V_n such that it is continuously decreasing to zero in the bands of width 1 but still keeps the discontinuous points in u. Outside V_n we define the process $Z_{n,\theta}(\cdot)$=0 . Now the process $Z_{n,\theta_0}(\cdot)$ is defined on the whole real line for all n, and the realizations of the process $Z_{n,\theta_0}(\cdot)$ belong to the space $\mathbf{D}_0(\mathbb{R})$ with probability 1.

We set for $z \in \mathbf{D}_0(\mathbb{R})$,

$$\begin{aligned}\Delta_h^l(z) &= \sup_{u,u',u''\in\delta_l}\left[\min\left\{\left|z(u') - z(u)\right|, \left|z(u'') - z(u)\right|\right\}\right]\\ &\quad + \sup_{l\le u\le l+h}\left|z(u) - z(l)\right| + \sup_{l+1-h\le u\le l+1}\left|z(u) - z(l+1)\right|.\end{aligned}$$

Here $l>0$ and $u,u^{'},u^{''}\in\delta_l$ means that $l\leq u-h\leq u^{'}<u<u^{''}\leq u+h\leq l+1$. The process $Z_{n,\theta_0}(\cdot)$ is considered over the interval $[l,l+1]$. We begin with the condition (2.5) for the process $Z_{n,\theta_0}^{1/4}(\cdot)$ and first estimate the probability $\mathbf{P}_{\theta_0}^{(n)}\left(\Delta_h^l(Z_{n,\theta_0}^{1/4})>h^{1/8}\right)$.

The process $\ln Z_{\theta_0,n}(u)$ admits the following representation (see for example [40])

$$\begin{aligned}\ln Z_{\theta_0,n}(u) &= \sum_{j=1}^{n}\sum_{i_j=1}^{m_j}\ln\frac{\lambda(\theta_0+\frac{u}{n},t_{ij})}{\lambda(\theta_0,t_{ij})}\\ &- n\int_0^{\tau}\Big(\lambda(\theta_0+\frac{u}{n},t)-\lambda(\theta_0,t)\Big)\,dt. \qquad (2.6)\end{aligned}$$

The pure jump component of the $\ln Z_{\theta_0,n}(\cdot)$ is given by

$$\sum_{j=1}^{n}\sum_{i_j=1}^{m_j}\ln\lambda(\theta_0+\frac{\cdot}{n},t_{ij}),$$

where t_{ij} are the jump times of the process X_j. When there is no event of the observed process on $[0,\tau]$ then we put $\sum_{i_j=1}^{m_j}\ln\lambda(\theta_0+\frac{\cdot}{n},t_{ij})=0$. Thus the process $Z_{n,\theta_0}^{1/4}(\cdot)$ has its jumps along the lines $u_{ij}=(t_{ij}-\theta_0)n$ and $u_{ij}=(t_{ij}-\theta_0-\tau_0)n$ (where i,j are such that $u_{ij}\in V_n$). Let $\mathbb{D}$ be the event that on the interval $[l,l+1]$ there exist at least two jumps of the process $Z_{n,\theta_0}(u)$ such that the distance between them is less than $2h$. We denote by $\mathbb{D}_p$ the event that the process $Z_{n,\theta_0}(u)$ has at least p jumps on the interval $(u,u+h)$ and $(u+\tau_0,u+\tau_0+h)$.

To estimate the probability $\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D})$, we need the following lemma

Lemma 5 *Let the conditions C_0 be satisfied, then there exists a constant $C>0$ such that*

$$\sup_{\theta_0\in\mathbf{K}}\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}_1)\leq Ch,$$

and

$$\sup_{\theta_0\in\mathbf{K}}\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}_2)\leq C^2h^2.$$

This lemma will be proved in the next section. Now it is possible to estimate the probability of the event $\mathbb{D}$. Subdivide the interval $[l,l+1]$ into $M_1{=}\left[\frac{1}{h}\right]$ intervals $d_i=(u_i,u_{i+1})$ of length M^{-1} . Each interval of length h is contained in either one of the intervals d_i or in one of the intervals $d_i\cup d_{i+1}$. Hence

$$\mathbb{D}\;\subset\;\left(\left[\bigcup_{i=1}^{M}\mathbb{D}_2(d_i)\right]\right)\bigcup\left(\left[\bigcup_{i=1}^{M-1}\mathbb{D}_2(d_i\cup d_{i+1})\right]\right).$$

This means that

$$\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}) \quad \leq \quad \sum_{i=1}^{M} \mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}_2(d_i)) + \sum_{i=1}^{M-1} \mathbf{P}_{\theta_0}^{(n)} \left\{ \mathbb{D}_2(d_i \cup d_{i+1}) \right\}.$$

Therefore when the event $\mathbb{D}$ occurs, then by the Lemma 5 we obtain the following estimate

$$\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}) \quad \leq \quad MCh^2 \leq Ch.$$

Evaluation of $\mathbb{C}_h$: Now we estimate the probability of the event:

$$\mathbb{C}_h = \Bigg\{ u \in \delta_l : \\ \sup_{u^{'}, u^{"} \in \delta_l} \left[\min \left\{ \left| Z_{n,\theta_0}^{\frac{1}{4}}(u^{'}) - Z_{n,\theta_0}^{\frac{1}{4}}(u) \right|, \left| Z_{n,\theta_0}^{\frac{1}{4}}(u^{"}) - Z_{n,\theta_0}^{\frac{1}{4}}(u) \right| \right\} \geq h^{1/8} \right] \Bigg\}.$$

If the event $\mathbb{D}^c$ occurs then each interval $(u-h, u+h)$ contains at most one point of discontinuity of the function $Z_{n,\theta}^{1/4}(\cdot)$, so that this function is continuous either on the interval $(u-h, u)$ or on the interval $(u, u+h)$. Suppose that the point of discontinuity belongs to the interval $(u-h, u)$, then the function $Z_{n,\theta_0}^{1/4}(u)$ is continuously differentiable over $(u, u+h)$ and

$$Z_{n,\theta_0}^{\frac{1}{4}}(u) - Z_{n,\theta_0}^{\frac{1}{4}}(u^{"}) = \int_{u^{"}}^{u} \frac{\partial}{\partial s} Z_{n,\theta_0}^{\frac{1}{4}}(s) ds.$$

Further, we have

$$\begin{aligned} Z_{n,\theta_0}^{\frac{1}{4}}(s) \quad &= \quad \exp\Bigg\{ \frac{1}{4} \sum_{j=1}^{n} \int_0^{\tau} \ln\Big(\frac{\lambda_0 + \lambda_1(t) \mathbb{1}_{\{t \in \Delta_s\}}}{\lambda_0 + \lambda_1(t) \mathbb{1}_{\{t \in \Delta\}}} \Big) dX_j(t) - \\ &- \frac{n}{4} \int_0^{\tau} \left[\lambda_1(t) \mathbb{1}_{t \in \Delta_s} - \lambda_1(t) \mathbb{1}_{\{t \in \Delta\}} \right] dt \Bigg\}. \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial}{\partial s} Z_{n,\theta_0}^{\frac{1}{4}}(s) = \frac{1}{4} \left(\frac{\partial}{\partial s} \ln Z_{n,\theta_0}(s) \right) Z_{n,\theta_0}^{\frac{1}{4}}(s) \\ &= \Bigg(\frac{1}{4} \frac{\partial}{\partial s} \sum_{j=1}^{n} \int_0^{\tau} \ln\Big(\frac{\lambda_0 + \lambda_1(t) \mathbb{1}_{t \in \Delta_s}}{\lambda_0 + \lambda_1(t) \mathbb{1}_{t \in \Delta}} \Big) dX_j(t) \\ &- \frac{n}{4} \frac{\partial}{\partial s} \int_0^{\tau} \left(\lambda_1(t) \mathbb{1}_{\{t \in \Delta_s\}} - \lambda_1(t) \mathbb{1}_{t \in \Delta} \right) dt \Bigg) Z_{n,\theta_0}^{\frac{1}{4}}(s) \end{aligned}$$

$$= \left(\frac{1}{4} \frac{\partial}{\partial s} \sum_{j=1}^{n} \int_{\theta_0}^{\theta_0+\frac{s}{n}} \ln \frac{\lambda_0}{\lambda_0+\lambda_1(t)} dX_j(t) + \frac{n}{4} \frac{\partial}{\partial s} \int_{\theta_0}^{\theta_0+\frac{s}{n}} \lambda_1(t)dt \right) Z_{n,\theta_0}^{\frac{1}{4}}(s)$$
$$+ \left(\frac{1}{4} \frac{\partial}{\partial s} \sum_{j=1}^{n} \int_{\theta_0+\tau_0}^{\theta_0+\frac{s}{n}+\tau_0} \ln \frac{\lambda_0+\lambda_1(t)}{\lambda_0} dX_j(t) - \frac{n}{4} \frac{\partial}{\partial s} \int_{\theta_0+\tau_0}^{\theta_0+\frac{s}{n}+\tau_0} \lambda_1(t)dt \right) Z_{n,\theta_0}^{\frac{1}{4}}(s)$$

$$= \left(\frac{1}{4} \frac{\partial}{\partial s} \sum_{j=1}^{n} \sum_{i_j=1}^{m_j} \ln \frac{\lambda_0}{\lambda_0+\lambda_1(t_{ij})} + \frac{n}{4} \frac{\partial}{\partial s} \int_{\theta_0}^{\theta_0+\frac{s}{n}} \lambda_1(t)dt \right) Z_{n,\theta_0}^{\frac{1}{4}}(s)$$
$$+ \left(\frac{1}{4} \frac{\partial}{\partial s} \sum_{j=1}^{n} \sum_{i_j=1}^{m_j} \ln \frac{\lambda_0+\lambda_1(t_{ij})}{\lambda_0} - \frac{n}{4} \frac{\partial}{\partial s} \int_{\theta_0+\tau_0}^{\theta_0+\frac{s}{n}+\tau_0} \lambda_1(t)dt \right) Z_{n,\theta_0}^{\frac{1}{4}}(s)$$

where $\left\{\theta_0 \leq t_{ij} \leq \theta_0 + \frac{s}{n}\right\}$ or $\left\{\theta_0 + \tau_0 \leq t_{ij} \leq \theta_0 + \frac{s}{n} + \tau_0\right\}$.

$$\begin{aligned}
\frac{\partial}{\partial s} Z_{n,\theta_0}^{\frac{1}{4}}(s) &= \left(\frac{n}{4} \frac{\partial}{\partial s} \int_{\theta_0}^{\theta_0+\frac{s}{n}} \lambda_1(t)dt - \frac{n}{4} \frac{\partial}{\partial s} \int_{\theta_0+\tau_0}^{\theta_0+\frac{s}{n}+\tau_0} \lambda_1(t)dt \right) Z_{n,\theta_0}^{\frac{1}{4}}(s) \\
&= \left(\frac{n}{4} \lambda_1(\theta_0 + \frac{s}{n}) \frac{1}{n} - \frac{n}{4} \lambda_1(\theta_0 + \frac{s}{n} + \tau_0) \frac{1}{n} \right) Z_{n,\theta_0}^{\frac{1}{4}}(s) \\
&= \left(\frac{1}{4} \left(\lambda_1(\theta_0 + \frac{s}{n}) - \lambda_1(\theta_0 + \frac{s}{n} + \tau_0) \right) \right) Z_{n,\theta_0}^{\frac{1}{4}}(s).
\end{aligned}$$

Then

$$\begin{aligned}
\left| \frac{\partial}{\partial s} Z_{n,\theta_0}^{\frac{1}{4}}(s) \right| &= \left| \left(\frac{1}{4} \left(\lambda_1(\theta_0 + \frac{s}{n}) - \lambda_1(\theta_0 + \frac{s}{n} + \tau_0) \right) \right) Z_{n,\theta_0}^{\frac{1}{4}}(s) \right| \\
&\leq \left(\frac{1}{4} \left(\left| \lambda_1(\theta_0 + \frac{s}{n}) \right| + \left| \lambda_1(\theta_0 + \frac{s}{n} + \tau_0) \right| \right) \right) Z_{n,\theta_0}^{\frac{1}{4}}(s) \\
&\leq \frac{C}{4} Z_{n,\theta_0}^{\frac{1}{4}}(s) \leq C Z_{n,\theta_0}^{\frac{1}{4}}(s);
\end{aligned}$$

and

$$\sup_{u \leq u" \leq u+h} \left| Z_{n,\theta_0}^{\frac{1}{4}}(u) - Z_{n,\theta_0}^{\frac{1}{4}}(u") \right| \leq C \int_{u"}^{u+h} Z_{n,\theta_0}^{\frac{1}{4}}(s)ds.$$

Introduce the process:

$$Y_n(u) = \int_l^u Z_{n,\theta_0}^{\frac{1}{4}}(s)ds. \tag{2.7}$$

For $\omega \in \mathbb{D}^c$ we obtain the inequality:

$$\sup_{u \in \delta_l} \sup_{u \leq u'' \leq u+h} \left| Z_{n,\theta_0}^{\frac{1}{4}}(u) - Z_{n,\theta_0}^{\frac{1}{4}}(u'') \right| \leq C \sup_{\left|u-u'\right|<h} \left| Y_n(u) - Y_n(u') \right|.$$

Indeed

$$\begin{aligned}\mathbf{E}_{\theta_0} Y_n^2(u) &= \left(\left(\mathbf{E}_{\theta_0} Y_n^2(u)\right)^2\right)^{\frac{1}{2}} \leq (u-l)\,\mathbf{E}_{\theta_0}\left(\int_l^u Z_{n,\theta_0}^{\frac{1}{4}}(s)ds\right)\\ &\leq (u-l)\int_l^u \mathbf{E}_{\theta_0} Z_{n,\theta_0}^{\frac{1}{2}}(s)ds \leq C\end{aligned}$$

and

$$\mathbf{E}_{\theta_0}\left|Y_n(u)-Y_n(u')\right|^2 = \mathbf{E}_{\theta_0}\left(\int_{u'}^u Z_{n,\theta_0}^{\frac{1}{4}}(s)ds\right)^2 \leq C\left|u-u'\right|^2.$$

Then applying Theorem A.19 in [32] with $H(u) = C$, $L = 1$, $r = 2$, $m = 2$ and $h = u + u'$, we obtain

$$\mathbf{E}_{\theta_0}\left(\sup_{\left|u-u'\right|<h}\left|Y_n(u)-Y_n(u^{'})\right|\right) \leq B_0 C^{\frac{1}{2}} l h^{\frac{1}{2}}. \tag{2.8}$$

Now

$$\begin{aligned}\mathbf{P}_{\theta_0}^n(\mathbb{C}_h) &= \mathbf{P}_{\theta_0}^n(\mathbb{C}_h,\mathbb{D}) + \mathbf{P}_{\theta_0}^n(\mathbb{C}_h,\mathbb{D}^c)\\ &\leq \mathbf{P}_{\theta_0}^n(\mathbb{D}) + \mathbf{P}_{\theta_0}^n\left\{\sup_{u\in\delta_l}\sup_{u\leq u''\leq u+h}\left|Z_{n,\theta_0}^{\frac{1}{4}}(u)-Z_{n,\theta_0}^{\frac{1}{4}}(u^{''})\right| > h^{\frac{1}{8}},\mathbb{D}^c\right\}\\ &\leq Ch + \mathbf{P}_{\theta_0}^n\left\{\sup_{\left|u-u'\right|<h}\left|Y_n(u)-Y_n(u^{'})\right| \geq ch^{\frac{1}{8}}\right\}.\end{aligned}$$

By Markov inequality and relation(2.8) we obtain

$$\mathbf{P}_{\theta_0}^n(\mathbb{C}_h) \leq Ch + B_0 C^{\frac{1}{2}} l \frac{h^{\frac{1}{2}}}{ch^{\frac{1}{8}}};$$

hence

$$\mathbf{P}_{\theta_0}^n(\mathbb{C}_h) \leq Ch^{\frac{3}{8}}.$$

The others terms of the modulus $\Delta_h^l(z)$ can be estimated in a similar way. This gives us the estimate

$$\begin{aligned}\mathbf{P}_{\theta_0}^n(\Delta_h^l(Z_{n,\theta_0}^{\frac{1}{4}}) > h^{\frac{1}{8}}) &\leq \mathbf{P}_{\theta_0}^n(\mathbb{D}) + \mathbf{P}_{\theta_0}^n\left(\Delta_h^l(Z_{n,\theta_0}^{\frac{1}{4}}) > h^{\frac{1}{8}},\mathbb{D}^c\right)\\ &\leq Ch + Dh^{\frac{1}{8}} \leq \gamma h^{\frac{3}{8}}.\end{aligned} \tag{2.9}$$

To end the proof we need also the following lemma.

Lemma 6 *Let*

$$M_n = \sup_{|u|<L} Z^{\frac{3}{4}}_{n,\theta_0}(u),$$

then we have

$$\mathbf{P}^n_{\theta_0}\left\{M_n > h^{\frac{-1}{16}}\right\} \le \kappa h^{\frac{1}{128}}.$$

For the proof see ([32] page 270). For L sufficiently large we have

$$|Z_{n,\theta_0}(u_1) - Z_{n,\theta_0}(u_2)| \le \sup_{|u|<L} Z^{\frac{3}{4}}_{n,\theta_0}(u)\left|Z^{\frac{1}{4}}_{n,\theta_0}(u_1) - Z^{\frac{1}{4}}_{n,\theta_0}(u_2)\right|$$

and

$$\Delta^l_h(Z_{n,\theta_0}) \le \Delta^l_h(Z^{\frac{1}{4}}_{n,\theta_0})M_n.$$

Therefore from (2.9) and lemma 6 we have

$$\begin{aligned}\mathbf{P}^n_{\theta_0}\left\{\Delta^l_h(Z_{n,\theta_0}) > h^{\frac{1}{16}}\right\} &\le \mathbf{P}^n_{\theta_0}\left\{\Delta^l_h(Z^{\frac{1}{4}}_{n,\theta_0})M_n > h^{\frac{1}{16}}, M_n \le h^{\frac{-1}{16}}\right\}\\ &+ \mathbf{P}^n_{\theta_0}\left\{M_n > h^{\frac{-1}{16}}\right\} \le \mathbf{P}^n_{\theta}\left\{\Delta^l_h(Z^{\frac{1}{4}}_{n,\theta_0}) > h^{\frac{1}{8}}\right\} + \kappa h^{\frac{1}{128}}\\ &\le \gamma h^{\frac{3}{8}} + \kappa h^{\frac{1}{128}} \le \varrho h^{\frac{1}{128}}.\end{aligned}$$

Now we have all the necessary inequalities to check the second condition of the Lemma 4 following the Theorem 5.3.1 in [32]. Therefore, the consistency and the convergence in law of the MLE of Theorem 3 are proved.

For the convergence of the moments of the MLE, we need to verify the uniform integrability of $n\left|\left(\hat\theta_n - \theta\right)\right|^p$ that is

$$\sup_{\theta_0\in\mathbf{K}} \mathbf{E}^n_{\theta_0}\left|n\left(\hat\theta_n - \theta_0\right)\right|^p < C.$$

Here the constant $C > 0$ does not depend on n. This verification is based on the estimate of the large deviation of the random process $Z_{n,\theta}(u)$ for the values of $|u|$. This estimate is obtained in [41], Chapter 5.

Proof of lemma 5.

Recall that the jumps of the process $Z_{n,\theta_0}(u)$ are at the points $u_{ij} = (t_{ij} - \theta_0)n$ and $u_{ij} = (t_{ij} - \theta_0 - \tau_0)n$ for $j = 1\cdots n$. Therefore, the event $\{u_{ij} \in (u, u+h)\}$ is equivalent to the event $\left\{t_{ij}, t_{ij} - \tau_0 \in (\theta_0 + \frac{u}{n}, \theta + \frac{u+h}{n})\right\}$. The later, in turn, is equivalent to the event $\mathbb{D}_1$. Put $a_n = \theta_0 + \frac{u}{n}$, $b_n = \theta_0 + \tau_0 + \frac{u}{n}$. Let $B^{(j)}_{p1}$ and $B^{(j)}_{p2}$,

$p = 1, 2$ be the events that the process X_j has at least p jumps on the intervals $(a_n, a_n + \frac{h}{n})$ and $(b_n, b_n + \frac{h}{n})$. Thus

$$B_{11}^{(j)} \cap B_{12}^{(j)} = \emptyset \quad \text{and} \quad \mathbb{D}_1 \subset \cup_{j=1}^{n} \left(B_{11}^{(j)} \cup B_{12}^{(j)} \right).$$

Therefore we obtain

$$\begin{aligned}
\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}_1) &\leq \sum_{j=1}^{n} \mathbf{P}_{\theta_0}^{(n)} \left(\left\{ B_{11}^{(j)} \right\} \bigcup \left\{ B_{12}^{(j)} \right\} \right) \\
&= \sum_{j=1}^{n} \mathbf{P}_{\theta_0}^{(n)} \left\{ B_{11}^{(j)} \right\} + \sum_{j=1}^{n} \mathbf{P}_{\theta_0}^{(n)} \left\{ B_{12}^{(j)} \right\} \\
&= \sum_{j=1}^{n} \mathbf{P}_{\theta_0}^{(n)} \left\{ X_j(a_n + \frac{h}{n}) - X_j(a_n) \geq 1 \right\} \\
&\quad + \sum_{j=1}^{n} \mathbf{P}_{\theta_0}^{(n)} \left\{ X_j(b_n + \frac{h}{n}) - X_j(b_n) \geq 1 \right\} \\
&= n \left(1 - \mathbf{P}_{\theta_0}^{(n)} \left\{ X_1(a_n + \frac{h}{n}) - X_1(a_n) = 0 \right\} \right) \\
&\quad + n \left(1 - \mathbf{P}_{\theta_0}^{(n)} \left\{ X_1(b_n + \frac{h}{n}) - X_1(b_n) = 0 \right\} \right) \\
&= n \left(1 - \exp \left\{ - \int_{a_n}^{a_n + \frac{h}{n}} (\lambda_0 + \lambda_1(t) \mathbb{1}_{\{\theta_0 \leq t \leq \theta_0 + \tau_0\}}) dt \right\} \right) \\
&\quad + n \left(1 - \exp \left\{ - \int_{b_n}^{b_n + \frac{h}{n}} (\lambda_0 + \lambda_1(t) \mathbb{1}_{\{\theta_0 \leq t \leq \theta_0 + \tau_0\}}) dt \right\} \right) \\
&\leq n \left(\int_{a_n}^{a_n + \frac{h}{n}} (\lambda_0 + \lambda_1(t)) dt \right) + n \left(\int_{b_n}^{b_n + \frac{h}{n}} (\lambda_0 + \lambda_1(t)) dt \right) \\
&= \frac{nhL}{n} + \frac{nhL}{n} = 2hL = Ch.
\end{aligned}$$

Furthermore we have

$$B_{21}^{(j)} = \left\{ X_j(a_n + \frac{h}{n}) - X_j(a_n) \geq 2 \right\},$$

the event that X_j has more than one jump on the interval $(a_n, a_n + \frac{h}{n})$ and

$$B_{22}^{(j)} = \left\{ X_j(b_n + \frac{h}{n}) - X_j(b_n) \geq 2 \right\},$$

is defined as the same way on the interval $(b_n, b_n + \frac{h}{n})$. Thus

$$\mathbb{D}_2 \subset \left[\left(\bigcup_{j=1}^{n}\bigcup_{k=j+1}^{n} B_{11}^{(j)}\bigcap B_{11}^{(k)}\right)\bigcup\left(\bigcup_{j=1}^{n} B_{21}^{(j)}\right)\right] \bigcup\left[\left(\bigcup_{j=1}^{n}\bigcup_{k=j+1}^{n} B_{12}^{(j)}\bigcap B_{12}^{(k)}\right)\bigcup\left(\bigcup_{j=1}^{n} B_{22}^{(j)}\right)\right].$$

The increments of Poisson process are independents on the disjoints intervals. Therefore we obtain

$$\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}_2) \leq \sum_{j=1}^{n}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{21}^{(j)}\right\} + \sum_{j=1}^{n}\sum_{k=j+1}^{n}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{11}^{(j)}\right\}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{11}^{(k)}\right\} + \sum_{j=1}^{n}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{22}^{(j)}\right\} + \sum_{j=1}^{n}\sum_{k=j+1}^{n}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{12}^{(j)}\right\}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{12}^{(k)}\right\},$$

and

$$\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}_2) \leq \sum_{j=1}^{n}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{21}^{(j)}\right\} + \sum_{j=1}^{n}\sum_{k=j+1}^{n}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{11}^{(j)}\right\}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{11}^{(k)}\right\} + \sum_{j=1}^{n}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{22}^{(j)}\right\} + \sum_{j=1}^{n}\sum_{k=j+1}^{n}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{12}^{(j)}\right\}\mathbf{P}_{\theta_0}^{(n)}\left\{B_{12}^{(k)}\right\}.$$

Using the arguments developed in [40], we show that

$$\mathbf{P}_{\theta_0}^{(n)}\left\{B_{21}^{(j)}\right\} \leq C^2h^2/n$$

and

$$\mathbf{P}_{\theta_0}^{(n)}\left\{B_{22}^{(j)}\right\} \leq C^2h^2/n.$$

Hence

$$\sup_{\theta\in\mathbf{K}}\mathbf{P}_{\theta}^{(n)}(\mathbb{D}_2) \leq C^2h^2 \quad \square$$

2.5 Simulations

We suppose that the observations $X^{(n)} = (X_1, \ldots, X_n)$ are n independent inhomogeneous Poisson processes $X_j = \left\{X_j(t),\, 0 \leq t \leq 10\right\}$, $j = 1, \ldots, n$ with the same intensity function

$$\lambda(\theta, t) = 1 + 2t\mathbb{1}_{\{\theta\leq t\leq\theta+2\}}, \qquad 0 \leq t \leq \tau$$

with $\theta \in (1,6)$ and $\tau = 8$. The true value of the parameter is θ_0 =2. Then we have

$$\begin{aligned} L\left(\theta, X^{(n)}\right) &= \exp\left\{\sum_{j=1}^{n}\int_0^8 \ln\bigl(1+2t\mathbb{1}_{\{\theta\leq t\leq\theta+2\}}\bigr)dX_j(t) - 4n(\theta+1)\right\} \\ &= \exp\left\{\sum_{j=1}^{n}\sum_{\theta\leq t_j^i\leq\theta+2} \ln\left(1+2t_j^i\right) - 4n(\theta+1)\right\}, \end{aligned} \tag{2.10}$$

where $\{t_j^i\}_{j=1,\cdots,N_j}$ $(N_j = X_j(10))$ are the events of the process X_j with intensity function $\lambda\,(2,t)$. The second sum in (2.10) is equal to zero when there is no event of the observed process.

For the normalized likelihood ratio function, simple calculus give the following expression: for $u \geq 0$,

$$\ln Z_{2,n}(u) = \sum_{j=1}^{n}\int_2^{2+\frac{u}{n}} -\ln\left(1+2t\right)dX_j(t) + \sum_{j=1}^{n}\int_4^{4+\frac{u}{n}} \ln\left(1+2t\right)dX_j(t) - 4u.$$

For $u < 0$,

$$\ln Z_{2,n}(u) = \sum_{j=1}^{n}\int_{2+\frac{u}{n}}^{2} \ln\left(1+2t\right)dX_j(t) + \sum_{j=1}^{n}\int_{4+\frac{u}{n}}^{4} -\ln\left(1+2t\right)dX_j(t) - 4u.$$

An illustration is given (see figure 1.2) on the behavior of the MLE $\hat{\theta}_n$ for different values of n.

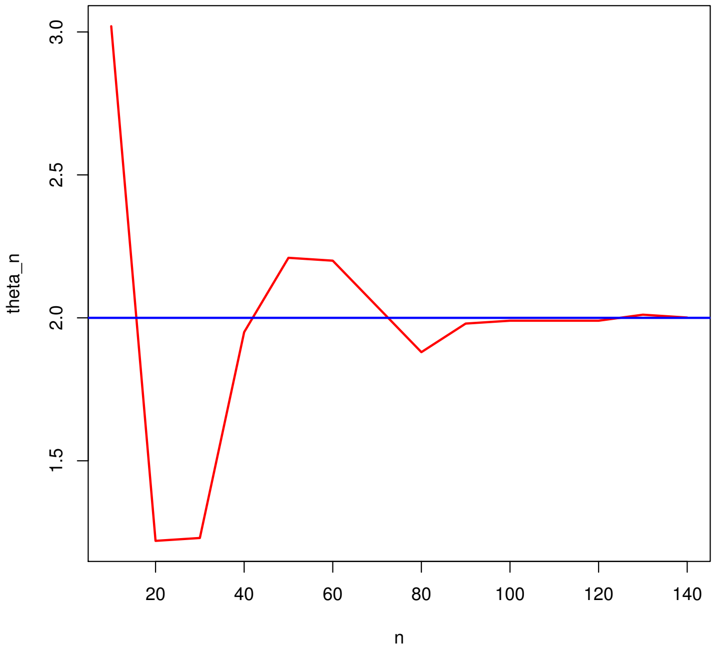

Figure 2.2: evolution of $\hat{\theta}_n$ with respect to n

Therefore the estimators $\hat{\theta}_n$ and $\widetilde{\theta}_n$ approach reasonably the true value $\theta_0 = 2$ for large values of n.

Denote $Z(u) = Z_{\theta_0}(u)$ and $\mathbf{E} = \mathbf{E}_{\theta_0}$. The limit likelihood ratio is

$$Z(u) = \exp\left\{\rho_1\ X^+(u) + \rho_2\ Y^+(u) - ru\right\}$$

for $u \geq 0$ and

$$Z(u) = \exp\left\{-\rho_1\ X^-(-u) - \rho_2\ Y^-(-u) - ru\right\}$$

for $u < 0$; where $\rho_1 = -\ln 5$, $\rho_2 = \ln 9$ and $r = 4$. Here $X^+(\cdot)$, $X^-(\cdot)$, $Y^+(\cdot)$ and $Y^-(\cdot)$ are independent Poisson processes on $\mathbb{R}_+$ of constant intensities 5, 1, 1 and 9 respectively. The limit random variables $\hat{u}$ and $\widetilde{u}$ satisfy

$$\max\left\{Z(\hat{u}-), Z(\hat{u}+)\right\} = \sup_{u\in\mathbb{R}} Z(u), \qquad \widetilde{u} = \frac{\int_{-\infty}^{+\infty} uZ(u)\ du}{\int_{-\infty}^{+\infty} Z(u)\ du}. \tag{2.11}$$

To visualize the properties of the sample path $Z(u)$,we conduct a simulation experiment. Thus we obtain the following figure.

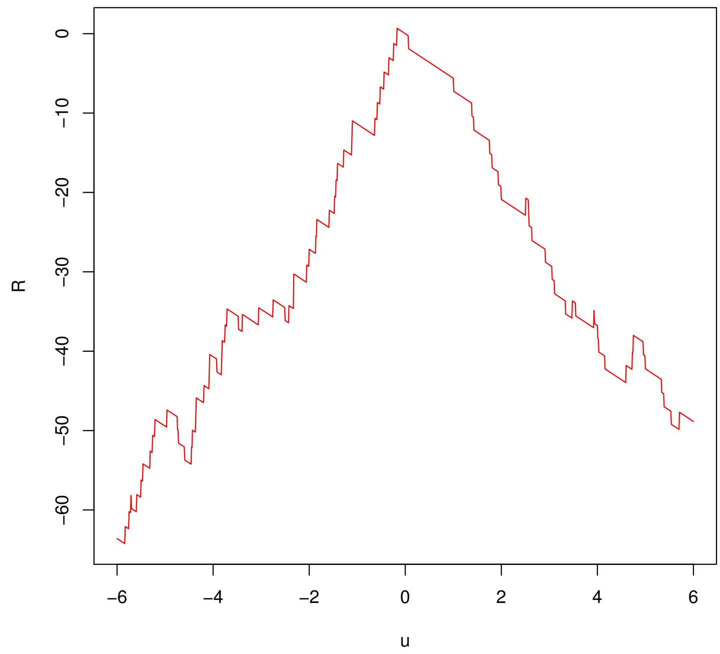

Figure 2.3: A sample path of the process $\ln Z(u)$

In the case of one discontinuity, the limit process $Z(u)$ contain a Poisson process $\Pi^{\pm}(u)$. Thus we remark that the argmax (the location of the point which maximize $Z(u)$) of the process $Z(u)$ must be a jump point of the Poisson process $\Pi^{\pm}(u)$. While, this technique can not be used in this present situation (model with two discontinuities). Then we can only simulate the processes separately and calculate this difference for several values of u on the real line. Therefore through the curve of $\ln Z(u)$ (figure 2.3), sometime we observe the peaks as in the case of one jump, sometime we observe the form of trapeze which corresponds the two jumps.

The maximum likelihood estimate $\hat{u}$ and the Bayesian estimator $\widetilde{u}$ satisfy the relation (2.11). To understand their behavior, we simulate the MLE for 10^4 times $(\hat{u}_1, \ldots, \hat{u}_{10000})$ and the BE for 10^4 times $(\widetilde{u}_1, \ldots, \widetilde{u}_{10000})$. Then the sample mean of $\hat{u}$ is 0.041 with standard deviation 1.11. While the sample mean of $\widetilde{u}$ is -0.0038 with standard deviation 0.753. It is interesting to see that these results of simulation are consistent with the theory that the empirical means

$$\sigma^2_{MLE} \approx \frac{1}{N}\sum_{l=1}^{N} \hat{u}_l^2 = 1.33 \quad and \quad \sigma^2_{BE} \approx \frac{1}{N}\sum_{l=1}^{N} \widetilde{u}_l^2 = 0.58.$$

satisfy

$$\sigma^2_{MLE} > \sigma^2_{BE}.$$

These values concur with the theoretical results that the Bayesian estimator outperforms the MLE. It concur also the i.i.d. case with one point of singularity (see [32] and [41])where it was mentioned that the Bayesian estimators are generally more efficient that the MLE estimators in Change-Point estimation.

Chapter 3

On multiple-change point estimation and hypothesis testing for Poisson process: case of zero jumps sum

3.1 Introduction

Among various issues in the structural change problem, estimation of the multiple change points is no doubt an interesting research topic. Presumably, one would like to estimate the locations of the change points when tests of structural change suggest that changes have occurred. Once change points are properly located, the original problem should be modified accordingly to provide better interpretation of data and more forecasts. In recent years a number of authors were concerned with these problems: in times series when the model is formulated in terms of latent discrete state variable that indicates the regime from which a particular observation has been drawn, see for example Chib [9], in regression estimation, for example, in Grégoire and Hamrouni [29], in a non parametric setting. Fields of application include econometrics, biostatistics, reliability and signal processing.

The general theory of parameter estimation in classical statistics is now well developed, see [1],[41] and the references therein. It is known that in the regular situation these estimators are consistent, asymptotically normal and asymptotically efficient. However the situation is different when the intensity function which characterizes the process is discontinuous and the corresponding families of measures are not locally asymptotically normal. In this case the limits of the likelihood ratios contain the Poisson processes, and the properties of the MLE and BE differs from the properties of these estimators as described in the regular case. Particularly the MLEs are no longer asymptotically efficient. The problems of change-point estimation for stochastic processes were considered by many authors. Let us mention here the works [41], [15]. Some studies are based on the model for Poisson process with fixed jump size. For those models, the limit distribution of the likelihood ratio is a log Poisson type process. Recall that problem of estimation of location of change-point were considered for the models of diffusion processes. The limiting likelihood ratios are a log Wiener processes see [32], [47].

Furthermore hypothesis testing in regular case for non homogeneous Poisson process has been considered by many authors, Kutoyants see [38] described locally asymptotically optimal tests, Leger [43] studied tests about constant versus monotonic intensity. Generally, if one fixes the alternative the power of any reasonable test tends toward one. Nevertheless it remains important to choose the best test. This can be done by examining their power to distinguish alternatives which are very close to the hypothesis $\mathscr{H}_0$, this corresponds to the most critical region of values of the alternative. The sequence defining the vicinity of the alternative can be chosen following the regularity of the problem. Therefore it allows us to obtain non degenerate limits for power functions and renders possible the comparison of the tests (Pitman approach [51]).

For our model, the normalized likelihood ratio converges to the difference of two log Poisson type processes. We use it to show that the Bayesian and maximum likelihood ratio estimators are consistent, converge in law and their moments converge also. For the test we give the properties of the Generalized Likelihood Ratio Test (GRLT) based on the maximum of the likelihood ratio function and the Wald Test (WT) based on the MLE. The behavior of these tests depends entirely on the properties of the normalized likelihood ratio process. Typically we consider the model of Poissonian observations in the situation were the intensity function have the following form:

$$\lambda\left(\theta,t\right)=\lambda_0+\lambda_1 \mathbb{1}_{\{\theta\leq t\leq\theta+\tau_0\}}$$

where λ_0, τ_0 and λ_1 are constants strictly positive. Thus it has two jumps located by the parameter θ. Note that such model is used in optical communication theory: the parameter (information) θ is transmitted over a channel with the help of the signal $\lambda_1 \mathbb{1}_{\{\theta\leq t\leq\theta+\tau_0\}}$ in the presence of the Poissonian noise of intensity λ_0 [40].

In Section 3.2 we describe the details of the model of observation, the asymptotic behavior of the likelihood ratio and the description of the limit process. The Section 3.3 is devoted to the parameter estimation. We describe the properties of estimators and the first proofs. Some complements of proof are given in Section 3.4.

It concerns the week convergence and the uniform integrability criterion. Section 3.5 consists of simulations. Finally, in Section 3.6 we present the asymptotic properties of GRLT and WT.

3.2 Change-point model with two jumps of zero sum

3.2.1 Preliminaries

We suppose that the observations $X^{(n)} = (X_1, \ldots, X_n)$ are n independent inhomogeneous Poisson processes $X_j = \left\{X_j(t),\, 0 \leq t \leq T\right\}$, $j = 1, \ldots, n$ with the same intensity function

$$\lambda(\theta, t) := \begin{cases} \lambda_0 & 0 \leq t < \theta \quad \text{or} \quad \theta + \tau_0 < t \leq \tau \\ \lambda_0 + \lambda_1 & \theta \leq t \leq \theta + \tau_0 \end{cases} \tag{3.1}$$

where

$$\theta \in \Theta = (\alpha, \beta), \tau = T - \tau_0, \quad \text{and} \quad 0 < \alpha < \beta < \beta + \tau_0 < \tau.$$

The constants λ_0, λ_1 and τ_0 are strictly positive. Recall that

$$\mathbf{E}_\theta X_j(t) = \Lambda\left(\theta, t\right) = \int_0^t \lambda\left(\theta, s\right)\, ds.$$

Then we have two jumps at points θ (a jump up) and $\theta + \tau_0$ (a jump down) and the distance between them is τ_0. Thus the processes X_j have a switching intensity $\lambda_0 + \lambda_1 1\!\!\mathrm{I}_{\{\theta \leq t \leq \theta + \tau_0\}}$ and a constant regime on each interval. The parameter θ is supposed to be unknown and we have to estimate it by the observations $X^{(n)}$. We have to describe properties of the MLE and the BE in the asymptotic of $n \longrightarrow \infty$.

3.2.2 Asymptotic behavior of the normalized likelihood ratio

Let $L\left(\theta, X^{(n)}\right)$ be the likelihood ratio of the model and $\mathbf{P}_\theta^{(n)}$ the measure induced in the space of observations by n realizations of the Poisson process with the intensity function $\lambda(\theta,t)$. Thus we have

$$L\left(\theta,X^{(n)}\right)=\exp\left\{\sum_{j=1}^{n}\int_0^{\tau}\ln\left(\lambda_0+\lambda_1 1\!\!1_{\{\theta\le t\le\theta+\tau_0\}}\right)dX_j(t)-n\int_0^{\tau}[\lambda(\theta,t)-1]dt\right\}$$
$$=\exp\left\{\sum_{j=1}^{n}\sum_{i_j=1}^{m_j}\ln\left(\lambda_0+\lambda_1 1\!\!1_{\left\{\theta\le t_{i_j}\le\theta+\tau_0\right\}}\right)-n\int_0^{\tau}[\lambda(\theta,t)-1]dt\right\}$$

where t_{i_j} are the jumps times of the process X_j and m_j the number of its points of discontinuity. Therefore each function $\lambda_0+\lambda_1 1\!\!1_{\left\{\theta\le t_{i_j}\le\theta+\tau_0\right\}}$ is discontinuous at points $\theta=t_{i_j}$ and $\theta=t_{i_j}-\tau_0$. The trajectories of the process $L\left(\cdot,X^{(n)}\right)$ have discontinuities at these points.

Let θ_0 be the true value of the parameter and put $\theta=\theta_0+\frac{u}{n}$. Given $\theta\in(\alpha,\beta)$; then we obtain $u\in\mathbb{U}_n=\Big((\alpha-\theta_0)n,(\beta-\theta_0)n\Big)$. Therefore we introduce the normalized likelihood ratio as follows

$$\begin{aligned}Z_{\theta_0,n}(u)&\equiv\frac{L\left(\theta_0+\frac{u}{n},X^{(n)}\right)}{L\left(\theta_0,X^{(n)}\right)}\\&=\exp\left\{\sum_{j=1}^{n}\int_0^{\tau}\ln\left(\frac{\lambda_0+\lambda_1 1\!\!1_{\left\{\theta_0+\frac{u}{n}\le t\le\theta_0+\frac{u}{n}+\tau_0\right\}}}{\lambda_0+\lambda_1 1\!\!1_{\{\theta_0\le t\le\theta_0+\tau_0\}}}\right)dX_j(t)-\right.\\&\left.-n\lambda_1\int_0^{\tau}\left(1\!\!1_{\left\{\theta_0+\frac{u}{n}\le t\le\theta_0+\frac{u}{n}+\tau_0\right\}}-1\!\!1_{\{\theta_0\le t\le\theta_0+\tau_0\}}\right)dt\right\}.\end{aligned}$$

If $u>0$ and $\frac{u}{n}<\tau_0$ then we can rewrite the process $Z_{\theta_0,n}(u)$ as follows

$$\begin{aligned}Z_{\theta_0,n}(u)&=\exp\left\{\sum_{j=1}^{n}\left(\int_{\Delta_u^1}\ln\frac{\lambda_0}{\lambda_0+\lambda_1}dX_j(t)+\int_{\Delta_u^2}\ln\frac{\lambda_0+\lambda_1}{\lambda_0}dX_j(t)\right)\right\}\\&=\exp\left\{\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\sum_{j=1}^{n}\left(\int_{\Delta_u^2}dX_j(t)-\int_{\Delta_u^1}dX_j(t)\right)\right\}.\end{aligned}\tag{3.2}$$

If $u>0$ and $\frac{u}{n}>\tau_0$ then

$$\begin{aligned}Z_{\theta_0,n}(u)&=\exp\left\{\sum_{j=1}^{n}\left(\int_{\Delta}\ln\frac{\lambda_0}{\lambda_0+\lambda_1}dX_j(t)+\int_{\Delta_u^3}\ln\frac{\lambda_0+\lambda_1}{\lambda_0}dX_j(t)\right)\right\}\\&=\exp\left\{\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\sum_{j=1}^{n}\left(\int_{\Delta_u^3}dX_j(t)-\int_{\Delta}dX_j(t)\right)\right\}\end{aligned}$$

where $\Delta_u^1 = [\theta, \theta + \frac{u}{n}]$, $\Delta_u^2 = [\theta + \tau_0, \theta + \tau_0 + \frac{u}{n}]$, $\Delta_u^3 = [\theta + \frac{u}{n}, \theta + \tau_0 + \frac{u}{n}]$ and $\Delta = [\theta, \theta + \tau_0]$.

Similarly for $u < 0$ and $\frac{|u|}{n} < \tau_0$ we obtain

$$Z_{\theta_0,n}(u) = \exp\left\{\ln \frac{\lambda_0 + \lambda_1}{\lambda_0} \sum_{j=1}^{n} \left(\int_{\Omega_u^1} dX_j(t) - \int_{\Omega_u^2} dX_j(t)\right)\right\};$$

if $\frac{|u|}{n} > \tau_0$ then

$$Z_{\theta_0,n}(u) = \exp\left\{\ln \frac{\lambda_0 + \lambda_1}{\lambda_0} \sum_{j=1}^{n} \left(\int_{\Omega_u^3} dX_j(t) - \int_{\Omega} dX_j(t)\right)\right\}.$$

where $\Omega_u^1 = [\theta + \frac{u}{n}, \theta]$, $\Omega_u^2 = [\theta + \tau_0 + \frac{u}{n}, \theta + \tau_0]$, $\Omega_u^3 = [\theta + \frac{u}{n}, \theta + \tau_0 + \frac{u}{n}]$ and $\Omega = [\theta, \theta + \tau_0]$.

We introduce the space $\mathbf{D}_0(\mathbb{R})$ of function $\varphi(u)$ without discontinuities of the second kind defined on $\mathbb{R}$ and such that $\lim\limits_{|u| \to +\infty} \varphi(u)=0$. We assume that all the function $\varphi(u) \in \mathbf{D}_0(\mathbb{R})$ are cadlàg.

Let φ_1 and φ_2 be two functions belonging to $\mathbf{D}_0(\mathbb{R})$. The Skorohod distance between $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ is defined as

$$d(\varphi_1, \varphi_2) = \inf_{\mu} \left[\sup_{\mathbb{R}} |\varphi_1(u) - \varphi_2(\mu(u))| + \sup_{\mathbb{R}} |u - \mu(u)|\right],$$

where the lower bound is taken over all the increasing continuous one-to-one mappings $\mu : \mathbb{R} \longrightarrow \mathbb{R}$. This metric space $(\mathbf{D}_0(\mathbb{R}), d(\cdot, \cdot))$ is complete and separable. Furthermore we denote $\Delta_h(z)$ the quantity defined by

$$\begin{aligned} \Delta_h(z) &= \sup_{u \in \mathbb{R}} \sup_{u-h \leq u' < u < u" \leq u+h} \left[\min\left\{\left|z(u') - z(u)\right|, \left|z(u") - z(u)\right|\right\}\right] \\ &+ \sup_{|u| > h^{-1}} |z(u)| . \end{aligned}$$

The conditions of weak convergence in $\mathbf{D}_0(\mathbb{R})$ are given in the following lemma

Lemma 7 *Let $z_{n,\theta}$, $n \in \mathbb{N}$, z_θ be random processes with realizations belonging to $\mathbf{D}_0(\mathbb{R})$ with probability 1. If, as $n \to +\infty$, the finite dimensional distributions of $z_{n,\theta}$ converge uniformly in $\theta \in \mathbf{K}$ to the finite dimensional distributions of z_θ and if for any $\epsilon > 0$*

$$\lim_{h \to 0} \sup_{n \in \mathbb{N}} \sup_{\theta \in \mathbf{K}} \mathbf{Q}_\theta^n \left\{\Delta_h(z_{n,\theta}) > \epsilon\right\} = 0. \tag{3.3}$$

Then for all functionals $\phi(\cdot) \in \mathbf{D}_0(\mathbb{R})$ the distribution of $\phi(z_{n,\theta})$ converges to the distribution of $\phi(z_\theta)$ uniformly in $\theta \in \mathbf{K}$, that is, z_{n,θ_0} converges weakly uniformly to z_θ.

Here and the sequel the set $\mathbf{K}$ denotes an arbitrary compact in Θ.

Introduce also the limit process $Z(\cdot)$

$$Z(u) := \begin{cases} \exp\Big\{\rho\left(Y^+(u) - X^+(u)\right)\Big\}, & u \geq 0, \\ \exp\Big\{\rho\left(X^-(-u) - Y^-(-u)\right)\Big\}, & u < 0 \end{cases}$$

where $\rho = \ln \frac{\lambda_0+\lambda_1}{\lambda_0}$. The Poisson processes $X^+(\cdot)$, $X^-(\cdot)$, $Y^+(\cdot)$ and $Y^-(\cdot)$ are independent on $\mathbb{R}_+$ such that $\mathbf{E}X^+(u) = \mathbf{E}Y^-(u) = (\lambda_0 + \lambda_1)u$ and $\mathbf{E}X^-(u) = \mathbf{E}Y^+(u) = \lambda_0 u$.

Remind that the process $Z_{n,\theta_0}(\cdot)$ has discontinuous trajectories defined on $\mathbb{U}_n$. Moreover correctly extending theses trajectories to the whole real line, on can consider that they belong to the Skorohod space $\mathbf{D}_0(\mathbb{R})$ (see paragraph 2.4.2). Under these conditions, the asymptotic behavior of the normalized likelihood ratio $Z_{n,\theta_0}(\cdot)$ is given by the following theorem.

Theorem 4 *Uniformly in $\theta_0 \in \mathbf{K}$, the process $Z_{\theta_0,n}(u)$ converge weakly in the space $D_0(\mathbb{R})$ to the process $Z_{\theta_0}(u)$.*

The proof of this theorem consists in checking the criterion of weak convergence. For this, we follows the methods and ideas used in [32] (see chapiters 5.3 and 5.4) and establish several lemmas. More precisely, the weak convergence in $\mathbf{D}_0(\mathbb{R})$ of the $Z_{\theta_0,n}(u)$ to the process $Z_{\theta_0}(u)$ follows from Theorem 5.4.2 of [40]. To check it we need the following lemmas

Lemma 8 *The finite dimensional distributions of the process $Z_{\theta_0,n}(u)$ converge to the finite dimensional distributions of the process $Z_{\theta_0}(u)$ and this convergence is uniform with respect to $\theta_0 \in \mathbf{K}$.*

Proof. Suppose that $u > 0$ (the other case can be treated in a similar way) and put $u = u_*$ (with fixed $u_* > 0$). Denote Δ and Δ_{u_*} the intervals defined by $\Delta = \{\theta_0 \leq t \leq \theta_0 + \tau_0\}$ and $\Delta_{u_*} = \left\{\theta_0 + \frac{u_*}{n} \leq t \leq \theta_0 + \tau_0 + \frac{u_*}{n}\right\}$ respectively. We can write

$$\ln Z_{\theta_0,n}(u_*) = \sum_{j=1}^{n} \int_0^\tau \ln\Big(\frac{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta_{u_*}\}}}{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}}\Big) dX_j(t) - n\int_0^\tau \lambda_1 \left[1\!\!1_{\{t\in\Delta_{u_*}\}} - 1\!\!1_{\{t\in\Delta\}}\right] dt.$$

Put

$$A_n = \sum_{j=1}^{n} \int_0^{\tau} \ln\left(\frac{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta_{u_*}\}}}{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}}\right) dX_j(t), \quad B_n = n\int_0^{\tau} \lambda_1 \left[1\!\!1_{\{t\in\Delta_{u_*}\}} - 1\!\!1_{\{t\in\Delta\}}\right] dt.$$

The characteristic function of $\ln Z_{\theta_0,n}(u_*)$ is calculated as follows

$$\Phi_n(y) = \mathbf{E}_{\theta_0} \exp\left(iy \ln Z_{\theta_0,n}(u_*)\right) = \exp\left(-iyB_n\right) \mathbf{E}_{\theta_0} \exp\left(iyA_n\right).$$

We remark that A_n is a sum of independents random variables, thus

$$\begin{aligned}&\mathbf{E}_{\theta_0} \exp\left(iyA_n\right)\\ &\quad = \exp\left\{n\int_0^{\tau}\left[\exp\left(iy\ln\frac{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta_{u_*}\}}}{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}}\right) - 1\right]\left(\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}\right) dt\right\}.\end{aligned}$$

As u_* is fixed, then $\frac{u_*}{n} \to 0$ for large value of n. Therefore we have only $\tau_0 > \frac{u_*}{n}$ and the interval can be expressed as the sum of five intervals $[0,\theta_0]$, $[\theta_0, \theta_0 + \frac{u_*}{n}]$,$[\theta_0 + \frac{u_*}{n}, \theta_0 + \tau_0]$, $[\theta_0 + \tau_0, \theta_0 + \frac{u_*}{n} + \tau_0]$ and $[\theta_0 + \frac{u_*}{n} + \tau_0, \tau]$.

The calculation of $\ln \mathbf{E}_{\theta_0} \exp\{iyA_n\}$ gives

$$\begin{aligned}\ln \mathbf{E}_{\theta_0} e^{iyA_n} &= n\int_{\theta_0}^{\theta_0+\frac{u_*}{n}}\left[\exp\left(iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1}\right) - 1\right](\lambda_0+\lambda_1)\, dt\\ &\quad + n\int_{\theta_0+\tau_0}^{\theta_0+\frac{u_*}{n}+\tau_0}\left[\exp\left(iy\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\right) - 1\right]\lambda_0 dt\end{aligned}$$

$$\begin{aligned}B_n &= n\lambda_1\int_0^{\tau}\left(1\!\!1_{\{t\in\Delta_{u_*}\}} - 1\!\!1_{\{t\in\Delta\}}\right) dt = -n\lambda_1\int_{\theta_0}^{\theta_0+\frac{u_*}{n}} dt + n\lambda_1\int_{\theta_0+\tau_0}^{\theta+\frac{u_*}{n}+\tau_0} dt\\ &= -\lambda_1 u_* + \lambda_1 u_* = 0.\end{aligned}$$

Then we obtain

$$\begin{aligned}\Phi_n(y) \quad \to \quad &\exp\left\{u_*\left[\exp\left(iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1}\right) - 1\right](\lambda_0+\lambda_1)\right.\\ &\left.+\left[u_*\exp\left(iy\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\right) - 1\right]\lambda_0\right\}.\end{aligned} \tag{3.4}$$

To end this proof we will verify that for $u_* > 0$, the characteristic function of $Z_{\theta_0}(u_*)$ coincides with the limit of $\Phi_n(y)$ i.e. the relation (3.4).
Indeed, for $u_* > 0$ we have

$$\begin{aligned}\Phi(y) &= \mathbf{E}_{\theta_0} e^{i\,y\ln Z_{\theta_0}(u_*)}\\ &= \mathbf{E}_{\theta_0}\exp\left\{i\,y\ln\left(\frac{\lambda_0}{\lambda_0+\lambda_1}\right)X^+(u_*) iy\ln\left(\frac{\lambda_0+\lambda_1}{\lambda_0}\right)Y^+(u_*)\right\};\end{aligned}$$

Due to the independency of the Poisson processes $X^+(\cdot)$, $Y^+(\cdot)$, we have

$$\Phi(y) = \mathbf{E}_{\theta_0} \exp\left\{iy \ln\left(\frac{\lambda_0}{\lambda_0+\lambda_1}\right) X^+(u_*)\right\} \mathbf{E}_\theta \exp\left\{iy \ln\left(\frac{\lambda_0+\lambda_1}{\lambda_0}\right) Y^+(u_*)\right\}.$$

Therefore we obtain the following calculations

$$\begin{aligned}
&\mathbf{E}_{\theta_0} \exp\left\{iy \ln\left(\frac{\lambda_0}{\lambda_0+\lambda_1}\right) X^+(u_*)\right\} \\
&\qquad = \exp\left\{\int_0^{u_*} \left[\exp\left(iy \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right) - 1\right] (\lambda_0+\lambda_1)\, dt\right\} \\
&\qquad = \exp\left\{u_* \left[\exp\left(iy \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right) - 1\right] (\lambda_0+\lambda_1)\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}_{\theta_0} \exp\left\{iy \ln\left(\frac{\lambda_0+\lambda_1}{\lambda_0}\right) Y^+(u_*)\right\} \\
&\qquad = \exp\left\{\int_0^{u_*} \left[\exp\left(iy \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right) - 1\right] \lambda_0 dt\right\} \\
&\qquad = \exp\left\{u_* \left[\exp\left(iy \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right) - 1\right] \lambda_0\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\Phi(y) &= \exp\left\{u_* \left[\exp\left(iy \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right) - 1\right] (\lambda_0+\lambda_1)\right\} \\
&\qquad \times \exp\left\{u_* \left[\exp\left(iy \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right) - 1\right] \lambda_0\right\}; \\
&= \exp\left\{u_* \left[\exp\left(iy \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right) - 1\right] (\lambda_0+\lambda_1)\right. \\
&\qquad \left. + u_* \left[\exp\left(iy \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right) - 1\right] \lambda_0\right\}.
\end{aligned}$$

This expression is equal to that in relation (3.4).

For $u < 0$, by a similar way we show that

$$\mathbf{E}_{\theta_0} \exp\{iy \ln Z_{\theta_0,n}(u)\} \to \exp\left\{-u\left[\exp\left(iy\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\right)-1\right]\lambda_0 - u\left[\exp\left(iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1}\right)-1\right](\lambda_0+\lambda_1)\right\}$$
$$= \mathbf{E}_{\theta_0}\exp\left\{iy\ln\frac{\lambda_0+\lambda_1}{\lambda_0}X^-((-u)) + iy\ln\frac{\lambda_0}{\lambda_0+\lambda_1}Y^-((-u))\right\}$$
$$= \mathbf{E}_{\theta_0}\exp\{iy\ln Z_{\theta_0}(u)\}.$$

Lemma 9 *There exists a constant $C > 0$ such that*

$$\sup_{\theta_0\in\mathbf{K}} \mathbf{E}_{\theta_0} \mid Z^{1/2}_{\theta_0,n}(u_1) - Z^{1/2}_{\theta_0,n}(u_2) \mid^2 \leq C \mid u_1 - u_2 \mid .$$

For all $n \in \mathbb{N}$, $u_1, u_2 \in \mathbb{U}_n$.

Proof. Suppose that $0 \leq u_1 \leq u_2$. First we treat the case $\frac{u_i}{n} < \tau_0$ for i=1,2. Thus according to the proposition 5 (Chapter 1), we have

$$\begin{aligned}
&\mathbf{E}_{\theta_0} \mid Z^{1/2}_{\theta_0,n}(u_1) - Z^{1/2}_{\theta_0,n}(u_2) \mid^2 \\
&\leq n\int_0^\tau \left[\sqrt{\frac{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_1}\}}}{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta\}}}} - \sqrt{\frac{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_2}\}}}{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta\}}}}\right]^2 \lambda(\theta_0,t)dt \\
&\leq n\int_0^\tau \frac{\left(\sqrt{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_1}\}}} - \sqrt{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_2}\}}}\right)^2}{\left(\sqrt{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta\}}}\right)^2}\lambda(\theta_0,t)dt \\
&\leq n\int_0^\tau \left(\sqrt{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_1}\}}} - \sqrt{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_2}\}}}\right)^2 dt \\
&= n\int_{\theta_0+\frac{u_1}{n}}^{\theta_0+\frac{u_2}{n}} \left(\sqrt{\lambda_0+\lambda_1}-\sqrt{\lambda_0}\right)^2 dt \\
&\quad + n\int_{\theta_0+\frac{u_1}{n}+\tau_0}^{\theta_0+\frac{u_2}{n}+\tau_0} \left(\sqrt{\lambda_0+\lambda_1}-\sqrt{\lambda_0}\right)^2 dt \\
&\leq C_1 \mid u_2-u_1 \mid + C_2 \mid u_2-u_1 \mid = C \mid u_2-u_1 \mid .
\end{aligned}$$

Hence

$$\mathbf{E}_{\theta_0} \mid Z^{1/2}_{\theta_0,n}(u_1) - Z^{1/2}_{\theta_0,n}(u_2) \mid^2 \leq C \mid u_2 - u_1 \mid$$

Consider also the case $\tau_0 \leq \frac{u_1}{n}$ and $\tau_0 + \frac{u_1}{n} < \frac{u_2}{n}$.

$$\begin{aligned} n\int_0^\tau &\left(\sqrt{\lambda_0 + \lambda_1 \mathbb{1}_{\{t\in\Delta_{u_1}\}}} - \sqrt{\lambda_0 + \lambda_1 \mathbb{1}_{\{t\in\Delta_{u_2}\}}}\right)^2 dt \\ &= n\int_{\theta_0+\frac{u_1}{n}}^{\theta_0+\frac{u_1}{n}+\tau_0} \left(\sqrt{\lambda_0+\lambda_1} - \sqrt{\lambda_0}\right)^2 dt \\ &\quad + n\int_{\theta_0+\frac{u_2}{n}}^{\theta_0+\frac{u_2}{n}+\tau_0} \left(\sqrt{\lambda_0+\lambda_1} - \sqrt{\lambda_0}\right)^2 dt \leq C\tau_0. \end{aligned}$$

As $\frac{u_1}{n} + \tau_0 \leq \frac{u_2}{n}$, we have $\tau_0 \leq \frac{u_2-u_1}{n}$. Hence

$$\begin{aligned} \mathbf{E}_{\theta_0} \mid Z^{1/2}_{\theta_0,n}(u_1) - Z^{1/2}_{\theta_0,n}(u_2) \mid^2 &\leq n\frac{C(u_2-u_1)}{n} \\ &\leq C(u_2-u_1) \end{aligned}$$

We obtain the similar results in the cases $\tau_0 < \frac{u_1}{n} < \frac{u_2}{n}$ and $\frac{u_1}{n} < \tau_0 < \frac{u_2}{n}$. Therefore for all $n \in \mathbb{N}$, $0 \leq u_1 \leq u_2$ and $\theta_0 \in \mathbf{K}$,

$$\mathbf{E}_{\theta_0} \mid Z^{1/2}_{\theta_0,n}(u_1) - Z^{1/2}_{\theta_0,n}(u_2) \mid^2 \leq C \mid u_2 - u_1 \mid .$$

For the other cases (say $u_2 < 0 < u_1$ etc.), the proofs can be carried out in a similar way.

Lemma 10 *There exists a constant $c > 0$ such that*

$$\sup_{\theta_0\in\mathbf{K}} \mathbf{E}_{\theta_0} Z^{1/2}_{\theta_0,n}(u) \leq e^{-c|u|}.$$

For all $n \in \mathbb{N}$, $u \in U_n$.

Proof. Suppose $u > 0$ (the case $u \leq 0$ can be treated in similar way). According to the proposition 5 (Chapter 1), we have

$$\mathbf{E}_\theta Z^{1/2}_{\theta,n}(u) = \exp\left\{\frac{-n}{2}\int_0^\tau \left(\sqrt{\frac{\lambda_0+\lambda_1\mathbb{1}_{\{t\in\Delta_{u*}\}}}{\lambda_0+\lambda_1\mathbb{1}_{\{t\in\Delta\}}}} - 1\right)^2 \left(\lambda_0 + \lambda_1 \mathbb{1}_{\{t\in\Delta\}}\right) dt\right\}.$$

If $\frac{u}{n} < \tau_0$, then we have

$$\int_0^\tau \left(\sqrt{\frac{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta_u\}}}{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}}} - 1\right)^2 \left(\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}\right) dt$$
$$= \int_0^\tau \left(\sqrt{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta_u\}}} - \sqrt{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}}\right)^2 dt$$
$$= \int_\theta^{\theta+\frac{u}{n}} \left(\sqrt{\lambda_0} - \sqrt{\lambda_0+\lambda_1}\right)^2 dt + \int_{\theta+\tau_0}^{\theta+\frac{u}{n}+\tau_0} \left(\sqrt{\lambda_0} - \sqrt{\lambda_0+\lambda_1}\right)^2 dt$$
$$= \int_{\theta_0}^{\theta_0+\frac{u}{n}} \frac{\lambda_1^2}{\left(\sqrt{\lambda_0}+\sqrt{\lambda_0+\lambda_1}\right)^2} dt + \int_{\theta_0+\tau_0}^{\theta_0+\frac{u}{n}+\tau_0} \frac{\lambda_1^2}{\left(\sqrt{\lambda_0}+\sqrt{\lambda_0+\lambda_1}\right)^2} dt$$
$$= c\frac{u}{n} + c\frac{u}{n} = 2c\frac{u}{n}.$$

Thus

$$\mathbf{E}_{\theta_0} Z_{\theta_0,n}^{1/2}(u) \quad \leq \quad e^{\frac{-n}{2}\frac{2uc}{n}} = e^{-c|u|}.$$

If $\frac{u}{n} > \tau_0$, then we have

$$\int_0^\tau \left(\sqrt{\frac{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta_u\}}}{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}}} - 1\right)^2 \left(\lambda_0 + \lambda_1(t) 1_{B_t}\right) dt$$
$$= \int_0^\tau \left(\sqrt{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta_u\}}} - \sqrt{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}}\right)^2 dt$$
$$= \int_\theta^{\theta+\tau_0} \left(\sqrt{\lambda_0} - \sqrt{\lambda_0+\lambda_1}\right)^2 dt + \int_{\theta+\frac{u}{n}}^{\theta+\frac{u}{n}+\tau_0} \left(\sqrt{\lambda_0} - \sqrt{\lambda_0+\lambda_1}\right)^2 dt$$
$$= \int_{\theta_0}^{\theta_0+\tau_0} \frac{\lambda_1^2}{\left(\sqrt{\lambda_0}+\sqrt{\lambda_0+\lambda_1}\right)^2} dt + \int_{\theta_0+\frac{u}{n}}^{\theta_0+\frac{u}{n}+\tau_0} \frac{\lambda_1^2}{\left(\sqrt{\lambda_0}+\sqrt{\lambda_0+\lambda_1}\right)^2} dt$$
$$= 2\tau_0 c_1.$$

Further $\theta = \theta_0 + \frac{u}{n}$ and $n = \frac{u}{\theta-\theta_0} \geq \frac{u}{\beta-\alpha}$. Therefore

$$\frac{n}{2}\int_0^\tau \left(\sqrt{\frac{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta_{u_*}\}}}{\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}}} - 1\right)^2 \left(\lambda_0 + \lambda_1 1\!\!1_{\{t\in\Delta\}}\right) dt \geq \frac{2u\tau_0 c_1}{2(\beta-\alpha)} = c \mid u \mid$$

$$\mathbf{E}_{\theta_0} Z_{\theta_0,n}^{1/2}(u) \leq e^{-c|u|} \qquad \Box$$

We need also to check the condition (3.3) (for $z_{n,\theta}=Z_{\theta_0}$). We set for $z \in \mathbb{D}_0(\mathbb{R})$,

$$\begin{aligned}\Delta_h^l(z) &= \sup_{u,u',u''\in\delta_l}\left[\min\left\{\left|z(u')\right|-|z(u)|,\left|z(u'')\right|-|z(u)|\right\}\right]\\ &+\sup_{l\le u\le l+h}|z(u)-z(l)|+\sup_{l+1-h\le u\le l+1}|z(u)-z(l+1)|.\end{aligned}$$

Here $l>0$ and $u,u',u''\in\delta_l$ means that $l\le u-h\le u'<u<u''\le u+h\le l+1$. Now we consider the process $Z_{n,\theta_0}(\cdot)$ over the interval $[l,l+1]$. Denote $\mathbb{D}$ the event that on the interval $[l,l+1]$ there exist at least two jumps of the process $Z_{n,\theta_0}(u)$ such that the distance between them is less than $2h$. Denote also $\mathbb{D}_p$ the event that the process $Z_{n,\theta_0}(u)$ has at least p jumps on the interval $(u,u+h)$ and $(u+\tau_0,u+\tau_0+h)$. Furthermore we represent the process $Z^{\frac14}_{n,\theta_0}(u)$ as the sum $Z^{\frac14}_{n,a}(u)+Z^{\frac14}_{n,s}(u)$, where $Z^{\frac14}_{n,a}(u)$ is absolutely continuous and $Z^{\frac14}_{n,s}(u)$ is the singular component of the function $Z^{\frac14}_{n,\theta_0}(u)$. Thus to verify the condition (3.3), we need the following lemmas.

Lemma 11 *There exist a positive constant $\gamma>0$ such that*

$$\mathbf{P}^n_{\theta_0}(\Delta_h^l(Z^{\frac14}_{n,\theta_0})>h^{\frac18})\le\gamma h^{\frac38}.$$

The proof is given in Section 3.4.

Lemma 12 *Let*

$$M_n=\sup_{|u|<L}Z^{\frac34}_{n,\theta_0}(u),$$

then we have

$$\mathbf{P}^n_{\theta_0}\left\{M_n>h^{\frac{-1}{16}}\right\}\le\kappa h^{\frac{1}{128}}.$$

For the proof see ([32] page 270). □

During the verification of Lemma 10, we obtain the following inequality which is needed to control the modulus of continuity $\Delta_h^l(Z^{\frac14}_{n,\theta_0})$.

$$\mathbf{P}^n_{\theta_0}\left(\sup_{|u|>D}Z_{n,\theta_0}(u)>e^{-bD}\right)\le Ce^{-bD}. \tag{3.5}$$

Let us check the condition (3.3). Indeed for L sufficiently large, we have the estimates.

$$|Z_{n,\theta_0}(u_1)-Z_{n,\theta_0}(u_2)|\le\sup_{|u|<L}Z^{\frac34}_{n,\theta_0}(u)\left|Z^{\frac14}_{n,\theta_0}(u_1)-Z^{\frac14}_{n,\theta_0}(u_2)\right|$$

and

$$\Delta_h^l(Z_{n,\theta_0}) \leq \Delta_h^l(Z_{n,\theta_0}^{\frac{1}{4}})M_n.$$

Therefore from Lemmas 11 and 12 we have

$$\begin{aligned}
\mathbf{P}_{\theta_0}^n\left\{\Delta_h^l(Z_{n,\theta_0}) > h^{\frac{1}{16}}\right\} &\leq \mathbf{P}_{\theta_0}^n\left\{\Delta_h^l(Z_{n,\theta_0}^{\frac{1}{4}}).M_n > h^{\frac{1}{16}}, M_n \leq h^{\frac{-1}{16}}\right\} \\
&+ \mathbf{P}_{\theta_0}^n\left\{M_n > h^{\frac{-1}{16}}\right\} \\
&\leq \mathbf{P}_{\theta}^n\left\{\Delta_h^l(Z_{n,\theta_0}^{\frac{1}{4}}) > h^{\frac{1}{8}}\right\} + \kappa h^{\frac{1}{128}} \\
&\leq \gamma h^{\frac{3}{8}} + \kappa h^{\frac{1}{128}} \leq \varrho h^{\frac{1}{128}}. \qquad (3.6)
\end{aligned}$$

Furthermore, we have also, for L sufficiently large

$$\mathbf{P}_{\theta_0}^n\left\{\Delta_h(Z_{n,\theta_0}) > 2h^{\frac{1}{16}}\right\} \leq \mathbf{P}_{\theta_0}^n\left\{\Delta_h^l(Z_{n,\theta_0}) > h^{\frac{1}{16}}\right\} + \mathbf{P}_{\theta_0}^n\left\{\sup_{|u|>\frac{L}{2}} Z_{n,\theta_0}(u) > h^{\frac{1}{16}}\right\}.$$

In view of inequalities (3.5) and (3.6), taking $\epsilon = 2h^{\frac{1}{16}}$, $D = \frac{L}{2}$ and $e^{-bD} = h^{\frac{1}{16}}$, we assert that the condition (3.3) is verified □

Note that the proof of Lemma 11 is obtained by the help of the following lemma.

Lemma 13 *There exists constants $C, D > 0$ such that the inequalities*

$$\mathbf{E}_{\theta_0}|Z_{n,a}^{\frac{1}{4}}(u+h) - Z_{n,a}^{\frac{1}{4}}(u)|^4 \leq Dh^4 \qquad (3.7)$$

$$\sup_{\theta_0\in\mathbf{K}} \mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}1) \leq Ch, \qquad (3.8)$$

and

$$\sup_{\theta_0\in\mathbf{K}} \mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}_2) \leq C^2h^2 \qquad (3.9)$$

hold.

The proof of this lemma is similar to that of Lemma 5 in Chapter 2.

3.2.3 Description of the limit process

Recall the limit process $Z(\cdot)$

$$Z(u) := \begin{cases} \exp\left\{\rho\left(Y^+(u) - X^+(u)\right)\right\}, & u \geq 0, \\ \exp\left\{\rho\left(X^-(-u) - Y^-(-u)\right)\right\}, & u < 0 \end{cases}$$

where $\rho = \ln \frac{\lambda_0+\lambda_1}{\lambda_0}$. The Poisson processes $X^+(\cdot)$, $X^-(\cdot)$, $Y^+(\cdot)$ and $Y^-(\cdot)$ are independent on $\mathbb{R}_+$ such that $\mathbf{E}X^+(u) = \mathbf{E}Y^-(u) = (\lambda_0 + \lambda_1)u$ and $\mathbf{E}X^-(u) = \mathbf{E}Y^+(u) = \lambda_0 u$.

On each part of $\mathbb{R}$ the process $Z(\cdot)$ is consists to the difference of two independent Poisson type processes of constant intensities. Further remaind that there is an intimate relation between the Poisson process and the exponential distribution. Indeed, let $\{N(t), t \geq 0\}$ be a Poisson process with intensity $\lambda^\star > 0$, then the intervals between jumps(events) are independent, exponentially distributed random variables with mean $\frac{1}{\lambda^\star}$. On the positive axis, we denote by T_i, T_i^{X+} and T_i^{Y+}, for $i = 1, 2, ...$, the inter arrival times of the processes $Z(\cdot)$, $X^+(\cdot)$ and $Y^+(\cdot)$ respectively. Therefore T_i^{X+} and T_i^{Y+} are independent exponential variables with parameters $\lambda_0 + \lambda_1$ and λ_0 respectively. The random variable T_i (see the figure 3.1) is defined as follows

$$T_i = \min(T_i^{X+}, T_i^{Y+})$$

and therefore is exponential with parameter $2\lambda_0 + \lambda_1$.

Denote by Y_i , $i = 1, 2, ..$, the random variable which characterize the nature of jumps of the process $Z(\cdot)$(hight jump or down jump). In fact if $Y_i = 1$, then we observe a hight jump with a probability equals to the intensity of $Y^+(\cdot)$ divided by the parameter of T_i i.e.

$$\mathbf{P}(Y_i = 1) = \frac{\lambda_0}{2\lambda_0 + \lambda_1}.$$

If $Y_i = -1$, then we observe a down jump with a probability equals to the intensity of $X^+(\cdot)$ divided by the parameter of T_i i.e.

$$\mathbf{P}(Y_i = -1) = \frac{\lambda_0 + \lambda_1}{2\lambda_0 + \lambda_1}.$$

Consequently the process $Z(\cdot)$ is compound Poisson process and we have the following representation

$$Z(u) := \begin{cases} \exp\left\{\rho \sum_{i=1}^{\prod^+(u)} Y_i\right\}, & u \geq 0, \\ \exp\left\{\rho \sum_{i=1}^{\prod^-(-u)} Y_i\right\}, & u < 0 \end{cases}$$

where $\prod^+(\cdot)$ and $\prod^-(\cdot)$ are Poisson processes of intensity $2\lambda_0 + \lambda_1$.

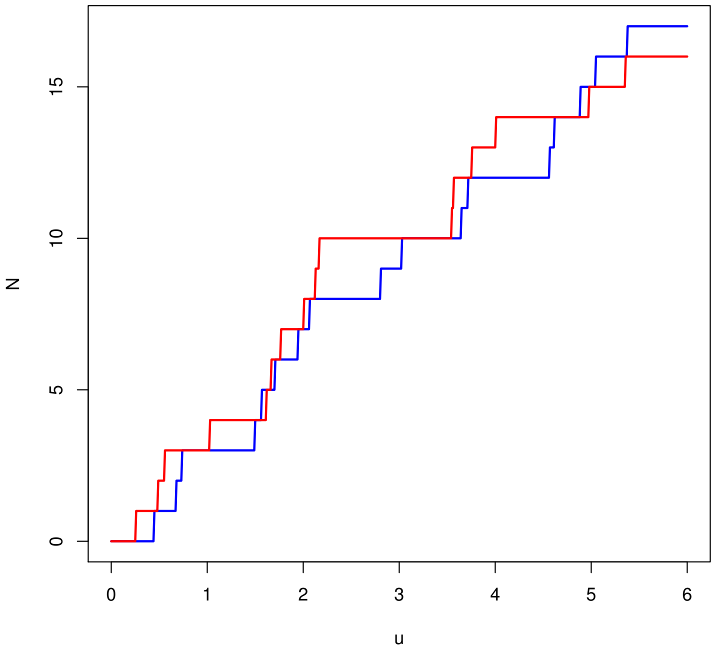

Figure 3.1: Realizations of the processes $X^+(\cdot)$ and $Y^+(\cdot)$
.

3.3 Parametric estimation

In this section we apply the convergence of normalized likelihood ratio obtained in section 3.2.2 to study the problem of estimation of the model (3.1). Thus we have to estimate θ from the observations $X^{(n)} = (X_1, \ldots, X_n)$ and to describe the asymptotic behavior of the estimators as $n \to \infty$. We consider both the maximum likelihood and Bayesian approaches. Recall that the likelihood function

$$\begin{aligned} L\left(\theta, X^{(n)}\right) &= \exp\Bigg\{\sum_{j=1}^{n}\int_0^{\tau} \ln\big(\lambda_0 + \lambda_1 \mathbb{1}_{\{\theta\le t\le\theta+\tau_0\}}\big)dX_j(t) \\ &\quad -n\int_0^{\tau}[\lambda_0 + \lambda_1 \mathbb{1}_{\{\theta\le t\le\theta+\tau_0\}} - 1]dt\Bigg\}. \end{aligned}$$

The maximum likelihood estimator (MLE) $\hat{\theta}_n$ is defined by the equation

$$L\left(\hat{\theta}_n, X^{(n)}\right) = \sup_{\theta\in\Theta} L\left(\theta, X^{(n)}\right). \tag{3.10}$$

The model has a piecewise constant regime. Therefore the likelihood ratio has constant realizations. The equation (3.10) has multiple solutions and any of them can be taken as the MLE. As we can see on the figure 3.1, the point of maximum of the likelihood function fill an interval $[a, b]$. We propose to call the MLE to be the middle of this interval (the interval that maximizes the likelihood). This choice will be justified in Section 3.5.

To apply the Bayesian approach, suppose that the unknown parameter is a random variable with a known prior density $p(\theta), \theta \in \Theta$, which is continuous and positive. Using the quadratic loss function, the Bayesian estimator (which minimizes the mean squares error) is the conditional mathematical expectation

$$\widetilde{\theta}_n = \int_\alpha^\beta \theta p(\theta/X^n) d\theta = \frac{\int_\alpha^\beta \theta p(\theta) L\left(\theta, X^{(n)}\right)\, d\theta}{\int_\alpha^\beta p(\theta) L\left(\theta, X^{(n)}\right)\, d\theta}.$$

We therefore recall properties of MLE and compare them with properties of Bayesian estimators. Recall the stochastic process

$$Z(u) := \begin{cases} \exp\Big\{\rho\left(Y^+(u) - X^+(u)\right)\Big\}, & u \geq 0, \\ \exp\Big\{\rho\left(X^-(-u) - Y^-(-u)\right)\Big\}, & u < 0 \end{cases}$$

where $\rho = \ln \frac{\lambda_0+\lambda_1}{\lambda_0}$. The Poisson processes $X^+(\cdot)$,$X^-(\cdot)$,$Y^+(\cdot)$ and $Y^-(\cdot)$ are independent on $\mathbb{R}_+$ such that $\mathbf{E}X^+(u) = \mathbf{E}Y^-(u) = (\lambda_0 + \lambda_1)u$ and $\mathbf{E}X^-(u) = \mathbf{E}Y^+(u) = \lambda_0 u$. The process $Z(u), u \in \mathbb{R}$ has piecewise constant realizations and the point $\hat{u}$ of the maximum of $Z(\cdot)$ is such that

$$Z(\hat{u}) = \sup_u Z(u);$$

where $\hat{u}$ is an intermediate point between $\hat{u}_l$ and $\hat{u}_r$ i.e. $\hat{u}_l < \hat{u} < \hat{u}_r$. If the highest interval is located on the positive axis (i.e. $Z(u), u \in \mathbb{R}^+$), then $\hat{u}_l$ and $\hat{u}_r$ will be the event of the processes $Y^+(\cdot)$ and $X^+(\cdot)$ respectively. If not, $\hat{u}_l$ and $\hat{u}_r$ will be the event of the processes $Y^-(\cdot)$ and $X^-(\cdot)$ respectively. The simulation of the process $Z(\cdot)$ is given in Section 3.5. The middle of the interval is given by the point

$$\hat{u} = \frac{\hat{u}_l + \hat{u}_r}{2}. \tag{3.11}$$

Introduce the random variable

$$\widetilde{u} = \frac{\int_{\mathbb{R}} uZ(u)du}{\int_{\mathbb{R}} Z(u)du}.$$

The main objective is to show that this random variable provides the limit distribution of the Bayesian estimator $n(\widetilde{\theta}_n - \theta_0) \Rightarrow \widetilde{u}$ and this property together with the convergence of moments of this estimator allows us to define the lower bound on the risk of all estimators. Also we will give the asymptotic properties of the MLE $\hat{u}$ and show that the limiting variance of $\widetilde{u}$ is smaller than $\hat{u}$. The main results are the following theorems.

Theorem 5 *The random variable $\widetilde{u}$ satisfy the following inequality*

$$\lim_{\delta\to 0} \underline{\lim}_{n\to+\infty} \inf_{\overline{\theta}_n} \sup_{|\theta-\theta_0|<\delta} n^2 \mathbf{E}_\theta \left(\overline{\theta}_n - \theta\right)^2 \geq \mathbf{E}_{\theta_0} \widetilde{u}^2 \tag{3.12}$$

where the inf is taken over all possible estimators $\overline{\theta}_n$ of the parameter θ

Using inequality (3.12), we give the following definition.

We say that an estimator $\overline{\theta}_n$ is asymptotically efficient if for all $\theta_0 \in \Theta$, the equality below holds

$$\lim_{\delta\to 0} \lim_{n\to+\infty} \sup_{|\theta-\theta_0|<\delta} n^2 \mathbf{E}_\theta \left(\overline{\theta}_n - \theta\right)^2 = \mathbf{E}_{\theta_0}(\widetilde{u}^2).$$

Theorem 6 *Uniformly in $\theta_0 \in \mathbf{K}$, the Baysian estimator $\widetilde{\theta}_n$ constructed by the observations X^n is consistent, the normalized difference $n(\widetilde{\theta}_n - \theta_0)$ converges in distribution:*

$$n(\widetilde{\theta}_n - \theta_0) \Rightarrow \widetilde{u}$$

and for any $p > 0$

$$\lim_{n\to+\infty} \mathbf{E}_{\theta_0} |n\left(\widetilde{\theta}_n - \theta_0\right)|^p = \mathbf{E}_{\theta_0} |\widetilde{u}|^p.$$

In particular $\widetilde{\theta}_n$ is asymptotically efficient.

Proof The proof of these theorems is based on the general result by Ibragimov and Khasminskii [32], Theorem 1.10.2. To apply it we study the normalized likelihood ratio process

$$Z_{\theta_0,n}(u) = \frac{L\left(\theta_0 + \frac{u}{n}, X^{(n)}\right)}{L\left(\theta_0, X^{(n)}\right)}, u \in \mathbb{U}_n = \Big((\alpha - \theta_0)n, (\beta - \theta_0)n\Big)$$

where θ_0 is the true value. Thus for $\theta = \theta_0 + \frac{u}{n}$, the Bayesian estimator can be written as

$$\begin{aligned}\widetilde{\theta}_n &= \frac{\int_\alpha^\beta \theta p(\theta) L\left(\theta, X^{(n)}\right)\, d\theta}{\int_\alpha^\beta p(\theta) L\left(\theta, X^{(n)}\right)\, d\theta} = \theta_0 + \frac{1}{n}\frac{\int_{\mathbb{U}_n} u p(\theta_0 + \frac{u}{n}) L\left(\theta_0 + \frac{u}{n}, X^{(n)}\right)\, du}{\int_{\mathbb{U}_n} p(\theta_0 + \frac{u}{n}) L\left(\theta_0 + \frac{u}{n}, X^{(n)}\right)\, du} \\ &= \theta_0 + \frac{1}{n}\frac{\int_{\mathbb{U}_n} u p(\theta_0 + \frac{u}{n}) \frac{L\left(\theta_0 + \frac{u}{n}, X^{(n)}\right)}{L\left(\theta_0, X^{(n)}\right)}\, du}{\int_{\mathbb{U}_n} p(\theta_0 + \frac{u}{n}) \frac{L\left(\theta_0 + \frac{u}{n}, X^{(n)}\right)}{L\left(\theta_0, X^{(n)}\right)}\, du} = \theta_0 + \frac{1}{n}\frac{\int_{\mathbb{U}_n} u p(\theta_0 + \frac{u}{n}) Z_{n,\theta_0}(u)\, du}{\int_{\mathbb{U}_n} p(\theta_0 + \frac{u}{n}) Z_{n,\theta_0}(u)\, du}.\end{aligned}$$

Therefore

$$n(\widetilde{\theta}_n - \theta_0) = \frac{\int_{\mathbb{U}_n} u p(\theta_0 + \frac{u}{n}) Z_{n,\theta_0}(u)\, du}{\int_{\mathbb{U}_n} p(\theta_0 + \frac{u}{n}) Z_{n,\theta_0}(u)\, du}.$$

In view of lemmas 8, 9,10 we can, referring to Theorem A.22 (see [32]), assert that the right term converge to

$$\frac{\int_{\mathbb{R}} uZ(u)du}{\int_{\mathbb{R}} uZ(u)du} \qquad i.e. \qquad n(\widetilde{\theta}_n - \theta_0) \Rightarrow \widetilde{u};$$

The consistency and the convergence of the moments of $\widetilde{\theta}_n$ also hold. Having these properties of Bayesian estimators we can cite Theorem 1.9.1 in [32] to provide the proof of Theorem 5.

Theorem 7 *Uniformly in $\theta_0 \in \mathbf{K}$, the maximum likelihood estimator $\hat{\theta}_n$ constructed by the observations X^n is consistent, the normalized difference $n(\hat{\theta}_n - \theta_0)$ converges in distribution:*

$$n(\hat{\theta}_n - \theta_0) \Rightarrow \hat{u}$$

and for any $p > 0$

$$\lim_{n\to+\infty} \mathbf{E}_{\theta_0} |n\left(\hat{\theta}_n - \theta_0\right)|^p = \mathbf{E}_{\theta_0}|\hat{u}|^p.$$

Proof. In view of the results of Section 3.2, the realizations of $Z_{n,\theta_0}(\cdot)$ belong to $D_0(\mathbb{R})$ with probability 1. Also its converges weakly to the process $Z(\cdot)$ in $D_0(\mathbb{R})$ (see Theorem 4). For any set $B \in \mathcal{B}(\mathbb{R})$, we define on $\mathbf{D}_0(\mathbb{R})$ the functionals $\Phi_B(\cdot)$ and $\Psi_B(\cdot)$ by

$$\Phi_B(\varphi) = \sup_{u\in B} \varphi(u) \quad \text{and} \quad \Psi_B(\varphi) = \sup_{u \in B^c} \varphi(u)$$

respectively. Thus $\Phi_B(\cdot)$ and $\Psi_B(\cdot)$ are continuous functionals in the sense of $d(\cdot,\cdot)$ (the Skorohod metric). Hence if we put $\hat{u}_n = n(\hat{\theta}_n - \theta_0)$ and $\Omega_B(Z_{n,\theta_0}) = \Phi_B(Z_{n,\theta_0}) - \Psi_B(Z_{n,\theta_0})$, then we have

$$\begin{aligned}\mathbf{P}_{\theta_0}^{(n)}\left(\hat{u}_n \in B\right) &= \mathbf{P}_{\theta_0}^{(n)}\left\{(\Phi_B(Z_{n,\theta}) > \Psi_B(Z_{n,\theta_0})\right\} \\ &= \mathbf{P}_{\theta_0}^{(n)}\left(\Omega_B(Z_{n,\theta_0}) > 0\right) \to \mathbf{P}_{\theta_0}\left(\Omega_B(Z_{\theta_0}) > 0\right) = \mathbf{P}_{\theta_0}\left(\hat{u} \in B\right).\end{aligned}$$

Thus
$$\mathbf{P}_{\theta_0}^{(n)}\left(\hat{u}_n \in B\right) \Longrightarrow \mathbf{P}_{\theta_0}\left(\hat{u} \in B\right) \tag{3.13}$$
which provide $n(\hat{\theta}_n - \theta_0) \Rightarrow \hat{u}$ and $\mathbf{P}_{\theta_0} - \lim_{n\to+\infty} \hat{\theta}_n = \theta_0$.

For any $p > 0$ the uniform integrability of $|n\left(\hat{\theta}_n - \theta_0\right)|^p$(see Section 3.4 for the proof)
$$\sup_n \mathbf{E}_{\theta_0}|n\left(\hat{\theta}_n - \theta_0\right)|^p < \infty. \tag{3.14}$$
holds.

Then this last relation together with (3.13) imply that
$$\lim_{n\to+\infty} \mathbf{E}_{\theta_0}|n\left(\hat{\theta}_n - \theta_0\right)|^p = \mathbf{E}_{\theta_0}|\hat{u}|^p.$$

3.4 Arguments for weak convergence criterion and uniform integrability

In this section we give the details of the proof for the weak convergence announced through the Section 3.2 and the uniform integrability of the random variable $|n\left(\hat{\theta}_n - \theta_0\right)|^p$. More precisely we give the proofs of Lemma 11 and relation (3.14).

Proof of Lemma 11. On an interval $[a, b]$ which does not contain any discontinuities of the process $Z_{n,\theta_0}^{\frac{1}{4}}(\cdot)$, the quantity $\Delta_h(Z_{n,\theta_0}^{\frac{1}{4}})$ coincides with $W_h(Z_{n,a}^{\frac{1}{4}})$ (the modulus of continuity of the process $Z_{n,a}^{\frac{1}{4}}(\cdot)$ on the interval $[l, l+1]$). As it is was proved in Theorem 1.A.19 in ([32]), the relation (3.7)of Lemma 13 implies the following bound
$$\mathbf{P}_\theta(W_h(Z_{n,a}^{\frac{1}{4}} > h^{\frac{1}{8}})) \leq Dh^{\frac{3}{8}}. \tag{3.15}$$
Therefore, in order to obtain from (3.15) the bound of Lemma 11, it is sufficient to show that on the interval $[l, l+1]$ there are not too many discontinuities of the process $Z_{n,\theta_0}(\cdot)$. Accordingly we have directly
$$\mathbf{P}_{\theta_0}^n(\Delta_h^l(Z_{n,\theta_0}^{\frac{1}{4}}) > h^{\frac{1}{8}}) \;\;\leq\;\; \mathbf{P}_{\theta_0}^n(\mathbb{D}) + \mathbf{P}_{\theta_0}^n\left(\Delta_h^l(Z_{n,\theta_0}^{\frac{1}{4}}) > h^{\frac{1}{8}}, \mathbb{D}^c\right). \tag{3.16}$$
Subdivide the interval $[l, l+1]$ into $M_1{=}\left[\frac{1}{h}\right]$ intervals $d_i = (u_i, u_{i+1})$ of length M^{-1}. Then each interval of length h is contained in either one of the intervals d_i or belongs to one of the intervals $d_i \cup d_{i+1}$. Therefore we obtain
$$\mathbb{D} \;\subset\; \left(\left[\bigcup_{i=1}^{M} \mathbb{D}_2(d_i)\right]\right) \bigcup \left(\left[\bigcup_{i=1}^{M-1} \mathbb{D}_2(d_i \cup d_{i+1})\right]\right)$$

and

$$\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}) \quad\leq\quad \sum_{i=1}^{M} \mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}_2(d_i)) + \sum_{i=1}^{M-1} \mathbf{P}_{\theta_0}^{(n)}\left\{\mathbb{D}_2(d_i \cup d_{i+1})\right\}. \tag{3.17}$$

Consequently when the event $\mathbb{D}$ occurs, by the relations (3.8) and (3.9) (of Lemma 13), we obtain the following estimate

$$\mathbf{P}_{\theta_0}^{(n)}(\mathbb{D}) \quad\leq\quad MCh^2 \leq Ch. \tag{3.18}$$

If the event $\mathbb{D}^c$ occurs, any interval of the form $(u-h, u+h)$ possesses at most one point of discontinuity of the function $Z_{n,\theta}^{1/4}(\cdot)$, so that this function is continuous either on the interval $(u-h, u)$ or on the interval $(u, u+h)$. Suppose that it belongs to the interval $(u-h, u)$, then the function $Z_{n,\theta_0}^{1/4}(u)$ is continuous on $(u, u+h)$. Then

$$Z_{n,\theta_0}^{\frac{1}{4}}(u) - Z_{n,\theta_0}^{\frac{1}{4}}(u^") = \int_{u^"}^{u} \frac{\partial}{\partial s} Z_{n,\theta_0}^{\frac{1}{4}}(s)ds.$$

Looking at the expression (3.2) we have

$$Z_{\theta_0,n}^{\frac{1}{4}}(s) = \exp\left\{\frac{1}{4}\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\sum_{j=1}^{n}\left(\int_{\theta_0+\tau_0}^{\theta+\tau_0+\frac{s}{n}} dX_j(t) - \int_{\theta_0}^{\theta+\frac{s}{n}} dX_j(t)\right)\right\}.$$

Therefore

$$\frac{\partial}{\partial s} Z_{n,\theta_0}^{\frac{1}{4}}(s) = \frac{1}{4}\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\sum_{j=1}^{n}\frac{\partial}{\partial s}\left(\int_{\theta_0+\tau_0}^{\theta+\tau_0+\frac{s}{n}} dX_j(t) - \int_{\theta_0}^{\theta+\frac{s}{n}} dX_j(t)\right) Z_{\theta_0,n}^{\frac{1}{4}}(s).$$

Thus

$$|\frac{\partial}{\partial s} Z_{n,\theta_0}^{\frac{1}{4}}(s)| \leq C Z_{n,\theta_0}^{\frac{1}{4}}(s)$$

and

$$\sup_{u \leq u^" \leq u+h} \left| Z_{n,\theta_0}^{\frac{1}{4}}(u) - Z_{n,\theta_0}^{\frac{1}{4}}(u^")\right| \leq C \int_{u^"}^{u+h} Z_{n,\theta_0}^{\frac{1}{4}}(s)ds.$$

The modulus of continuity of the process $Z_{n,a}^{\frac{1}{4}}(\cdot)$ is

$$\sup_{\left|u-u^{'}\right|<h} \left| Y_n(u) - Y_n(u^{'})\right|$$

where the process $Y_n(\cdot)$ is defined as follows

$$Y_n(u) = \int_l^u Z_{n,\theta_0}^{\frac{1}{4}}(s)ds. \tag{3.19}$$

For $\omega \in \mathbb{D}^c$ we obtain the inequality

$$\sup_{u\in\delta_l}\sup_{u\le u"\le u+h}\left|Z_{n,\theta_0}^{\frac{1}{4}}(u) - Z_{n,\theta_0}^{\frac{1}{4}}(u^")\right| \le C \sup_{\left|u-u'\right|<h}\left|Y_n(u) - Y_n(u')\right|.$$

Indeed

$$\begin{aligned}
\mathbf{E}_{\theta_0} Y_n^2(u) &= \left(\left(\mathbf{E}_{\theta_0} Y_n^2(u)\right)^2\right)^{\frac{1}{2}} \\
&\le (u-l)\,\mathbf{E}_{\theta_0}\left(\int_l^u Z_{n,\theta_0}^{\frac{1}{4}}(s)ds\right) \\
&\le (u-l)\int_l^u \mathbf{E}_{\theta_0} Z_{n,\theta_0}^{\frac{1}{2}}(s)ds \le C
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}_{\theta_0}\left|Y_n(u) - Y_n(u')\right|^2 &= \mathbf{E}_{\theta_0}\left(\int_{u'}^u Z_{n,\theta_0}^{\frac{1}{4}}(s)ds\right)^2 \\
&\le C\left|u-u'\right|^2.
\end{aligned}$$

Then applying Theorem A.19 in [32] with $H(u) = C$, $L = 1$, $r = 2$, $m = 2$ and $h = u + u'$, we obtain

$$\mathbf{E}_{\theta_0}\left(\sup_{\left|u-u'\right|<h}\left|Y_n(u) - Y_n(u')\right|\right) \le B_0 C^{\frac{1}{2}} l h^{\frac{1}{2}}. \tag{3.20}$$

Now introduce the set

$$\mathbb{C}_h = \Bigg\{u \in \delta_l : \sup_{u',u"\in\delta_l}\left[\min\left\{\left|Z_{n,\theta_0}^{\frac{1}{4}}(u') - Z_{n,\theta_0}^{\frac{1}{4}}(u)\right|, \left|Z_{n,\theta_0}^{\frac{1}{4}}(u^") - Z_{n,\theta_0}^{\frac{1}{4}}(u)\right|\right\} \ge h^{1/8}\right]\Bigg\}.$$

According to the calculations above, we obtain

$$\begin{aligned}
\mathbf{P}_{\theta_0}^n(\mathbb{C}_h) &= \mathbf{P}_{\theta_0}^n(\mathbb{C}_h, \mathbb{D}) + \mathbf{P}_{\theta_0}^n(\mathbb{C}_h, \mathbb{D}^c) \\
&\le \mathbf{P}_{\theta_0}^n(\mathbb{D}) + \mathbf{P}_{\theta_0}^n\left\{\sup_{u\in\delta_l}\sup_{u\le u"\le u+h}\left|Z_{n,\theta_0}^{\frac{1}{4}}(u) - Z_{n,\theta_0}^{\frac{1}{4}}(u^")\right| > h^{\frac{1}{8}}, \mathbb{D}^c\right\} \\
&\le Ch + \mathbf{P}_{\theta_0}^n\left\{\sup_{\left|u-u'\right|<h}\left|Y_n(u) - Y_n(u')\right| \ge ch^{\frac{1}{8}}\right\}.
\end{aligned}$$

By Markov inequality and (3.20) we obtain

$$\mathbf{P}^n_{\theta_0}(\mathbb{C}_h) \leq Ch + B_0 C^{\frac{1}{2}} l \frac{h^{\frac{1}{2}}}{ch^{\frac{1}{8}}};$$

hence

$$\mathbf{P}^n_{\theta_0}(\mathbb{C}_h) \leq Ch^{\frac{3}{8}}.$$

The others terms of the distance $\Delta^l_h(z)$ can be estimated in a similar way. This gives us the estimate

$$\begin{aligned}\mathbf{P}^n_{\theta_0}(\Delta^l_h(Z^{\frac{1}{4}}_{n,\theta_0}) > h^{\frac{1}{8}}) &\leq \mathbf{P}^n_{\theta_0}(\mathbb{D}) + \mathbf{P}^n_{\theta_0}\left(\Delta^l_h(Z^{\frac{1}{4}}_{n,\theta_0}) > h^{\frac{1}{8}}, \mathbb{D}^c\right) \\ &\leq Ch + Dh^{\frac{3}{8}} \leq \gamma h^{\frac{3}{8}} \qquad \square\end{aligned}$$

Uniform integrability: By a similar way of Lemma 12 we show that: there exist $L > 0$ such that for all $x > L$,

$$\mathbf{P}^n_{\theta_0}\left\{\sup_{|u|>x} Z_{n,\theta_0}(u) > e^{-kx}\right\} \leq \gamma e^{-kx} \tag{3.21}$$

where $k, \gamma > 0$ are constants.

Let $Y_n = n|\hat{\theta}_n - \theta_0|$. We have

$$\begin{aligned}\mathbf{E}_{\theta_0} Y_n^p &= \int_0^{+\infty} x^p d\mathbf{P}^n_{\theta_0}\{Y_n \leq x\} \\ &\leq \int_0^L x^p d\mathbf{P}^n_{\theta_0}\{Y_n \leq x\} - \int_L^{+\infty} x^p d\mathbf{P}^n_{\theta_0}\{Y_n > x\} \\ &\leq L^p - [x^p \mathbf{P}^n_{\theta_0}\{Y_n > x\}]_L^{+\infty} + p\int_L^{+\infty} x^{p-1}\mathbf{P}^n_{\theta_0}\{Y_n > x\}dx.\end{aligned}$$

Further

$$\begin{aligned}\mathbf{P}^n_{\theta_0}\{Y_n > x\} &= \mathbf{P}^n_{\theta_0}\{n|\hat{\theta}_n - \theta_0| > x\} \\ &= \mathbf{P}^n_{\theta_0}\left\{\sup_{u\in[-x,x]^c} Z_{n,\theta_0}(u) > \sup_{u\in[-x,x]} Z_{n,\theta_0}(u)\right\},\end{aligned}$$

we deduce that

$$\mathbf{P}^n_{\theta_0}\{Y_n > x\} \leq \mathbf{P}^n_{\theta_0}\left\{\sup_{|u|>x} Z_{n,\theta_0}(u) > Z_{n,\theta_0}(0)\right\} \leq \mathbf{P}^n_{\theta_0}\left\{\sup_{|u|>x} Z_{n,\theta_0}(u) > 1\right\}.$$

However

$$\mathbf{P}_{\theta_0}^n\left\{\sup_{|u|>x} Z_{n,\theta_0}(u) > 1\right\} \leq \mathbf{P}_{\theta_0}^n\left\{\sup_{|u|>x} Z_{n,\theta_0}(u) > e^{-kx}\right\}.$$

Therefore we obtain from (3.21)

$$\begin{aligned}\mathbf{E}_{\theta_0}|n\left(\hat{\theta}_n - \theta_0\right)|^p &\leq L^p + p\gamma\int_L^{+\infty} x^{p-1}e^{-kx}dx \\ &\leq C.\end{aligned}$$

Hence

$$\sup_n \mathbf{E}_{\theta_0}|n\left(\hat{\theta}_n - \theta_0\right)|^p < \infty.\Box$$

3.5 Simulations

Consider the intensity function $\lambda(\theta, t) = \lambda_0 + \lambda_1 1\!\!1_{\{\theta \leq t \leq \theta+2\}}, 0 \leq t \leq 10$ where $\lambda_0 = 1$, $\lambda_1 = 2$ and $\tau_0 = 2$. The true value of the parameter to estimate is θ_0 =2. Then we have

$$\begin{aligned}L\left(\theta, X^{(n)}\right) &= \exp\left\{\sum_{j=1}^n \int_0^{10} \ln\left(1 + 2 1\!\!1_{\{\theta\leq t\leq\theta+2\}}\right) dX_j(t) - 4n\right\} \\ &= \exp\left\{\ln(\lambda_0 + \lambda_1)\sum_{j=1}^n \sum_{\theta\leq t_j^i\leq\theta+2} -4n\right\},\end{aligned} \tag{3.22}$$

where $\{t_j^i\}_{j=1,\cdots,N_j}$ are the events of the process X_j with intensity function $\lambda(\theta_0, t)$. The second sum in (3.22) is equal to zero when there is no event of the observed process.

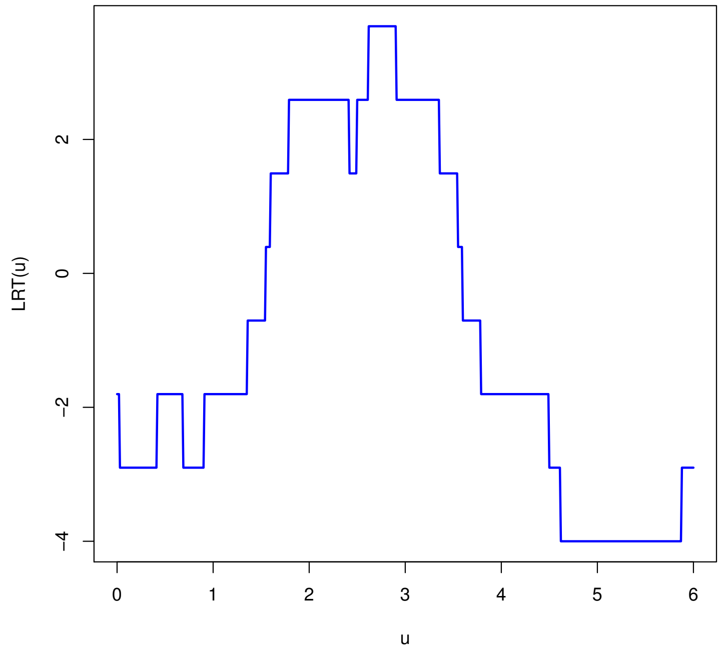

Figure 3.2: A realization of the process $L\left(\theta, X^{(n)}\right)$

Recall that the process $Z(\cdot)$ is defined as follows

$$Z(v) := \begin{cases} \exp\left\{\rho\left(Y^{+}(v) - X^{+}(v)\right)\right\} & v \geq 0 \\ \exp\left\{\rho\left(X^{-}(-v) - Y^{-}(-v)\right)\right\} & v < 0 \end{cases}$$

where $\rho = \ln \frac{\lambda_0+\lambda_1}{\lambda_0}$. The Poisson processes $X^{+}(\cdot)$,$X^{-}(\cdot)$,$Y^{+}(\cdot)$ and $Y^{-}(\cdot)$ are independent on $\mathbb{R}_{+}$ such that $\mathbf{E}X^{+}(v) = \mathbf{E}Y^{-}(v) = (\lambda_0 + \lambda_1)v$ and $\mathbf{E}X^{-}(v) = \mathbf{E}Y^{+}(v) = \lambda_0 v$.

The argmax $\hat{v}$ of Z can be obtained as follow:

$$\hat{v} = \rho\hat{u}$$

where

$$Z_1(\hat{u}) = \sup_u Z_1(u)$$

with

$$Z_1(u) := \begin{cases} \exp\Big\{Y^+(u) - X^+(u)\Big\} & u \geq 0 \\ \exp\Big\{X^-(-u) - Y^-(-u)\Big\} & u < 0. \end{cases}$$

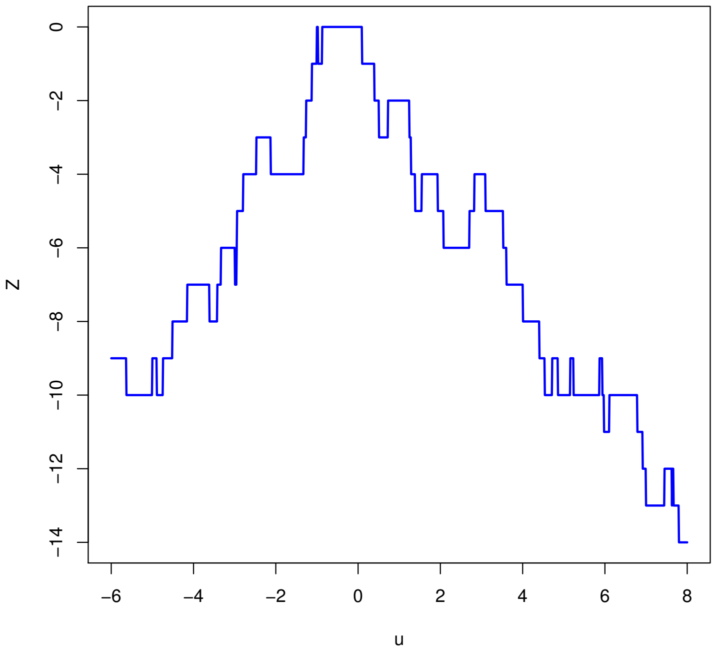

Figure 3.3: A realization of the process $\ln Z_1(u)$

As we can see over the figure, the highest interval $[\hat{u}_l, \hat{u}_r]$ of the process $Z_1(\cdot)$ can be on the positive, negative side of $\mathbb{R}$ or it can be $[-a^-, a^+]$, where a^- is the smallest event of the processes $X^-(\cdot)$ and $Y^-(\cdot)$. The event a^+ is also the smallest event of the processes $X^+(\cdot)$ and $Y^+(\cdot)$. We remark that this last situation arise if a^- and a^+ are the first event of the Poisson processes $Y^-(\cdot)$ and $X^+(\cdot)$ respectively. Also if $[\hat{u}_l, \hat{u}_r] = [\hat{u}_l^+, \hat{u}_r^+]$ (on the positive side of $\mathbb{R}$) then $\hat{u}_l^+$ and $\hat{u}_r^+$ are events of the Poisson processes $Y^+(\cdot)$ and $X^+(\cdot)$ respectively. Further if $[\hat{u}_l, \hat{u}_r] = [\hat{u}_l^-, \hat{u}_r^-]$ (on the negative side of $\mathbb{R}$) then $-\hat{u}_l^-$ and $-\hat{u}_r^-$ are events of the Poisson processes $Y^-(\cdot)$ and $X^-(\cdot)$ respectively. Thus our goal is to find the random variable $\hat{u}$ that realizes the maximum.

We get it by the following steps:

- First we simulate the process $Z_1(u)$ for $N = 10^4$ times. One construct the convex combination

$$\hat{u}_{\gamma,i} = \hat{u}_r^i + \delta(\hat{u}_l^i - \hat{u}_r^i)$$

where $[\hat{u}_l^i, \hat{u}_r^i]$ constitute the highest interval of the i^{th} replication of $Z_1(u)$ and $\gamma \in (0,1)$.

- Let

$$\sigma_N^2(\gamma) = \frac{1}{N}\sum_{i=1}^{N} \hat{u}_{\gamma,i}^2$$

then we determine the value of γ which realize the minimum i.e. the solution of the equation

$$\sigma_N^2(\gamma_0) = \min_{\gamma\in(0,1)} \sigma_N^2(\gamma) \tag{3.23}$$

For $N = 10^4$ we get $\gamma = 0.506$. Hence the MLE is $\hat{u} = \frac{\hat{u}_l+\hat{u}_r}{2}$, the middle of the the highest interval.

3.6 Hypothesis testing

3.6.1 Preliminaries

Recall that $X = (X_t, t \geq 0)$ is an inhomogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$, if $X_0 = 0$ and the increments of X on disjoints intervals are independent and distributed according to the Poisson law

$$\mathbf{P}\{X_t - X_s = k\} = \frac{\left(\int_s^t \lambda(u)du\right)^k}{k!} \exp\left\{-\int_s^t \lambda(u)du\right\}.$$

In this part we take the same model observations. Indeed we suppose that the observations $X^{(n)} = (X_1, \ldots, X_n)$ are n independent inhomogeneous Poisson processes $X_j = \left\{X_j(t),\, 0 \leq t \leq T\right\}$, $j = 1, \ldots, n$ with the same intensity function

$$\lambda(\theta, t) = \lambda_0 + \lambda_1 1\!\!1_{\{\theta \leq t \leq \theta+\tau_0\}}, \quad 0 \leq t \leq \tau.$$

with constants $\lambda_0, \lambda_1, \tau_0 > 0$. The parameter $\theta \in \Theta = [\theta_1, \beta)$ and

$$\tau = T - \tau_0, \quad 0 < \theta_1 < \beta < \beta + \tau_0 < \tau.$$

Under this condition we have the two jumps of the intensity function on the interval of observations for $\theta \in \Theta$. Recall that

$$\mathbf{E}_\theta X_j(t) = \Lambda(\theta, t) = \int_0^t \lambda(\theta, s)\, ds.$$

In practice the problem can have following conjuncture: Suppose that θ_1 is the first jump of the process. So, the change point(s) problem is twofold: one is to decide if the initial λ_0 regime is changed. In others words if the first jump is exceeded or not; which corresponds to the following schema of test

$$\mathscr{H}_0 \quad : \quad \theta = \theta_1,$$

and alternative

$$\mathscr{H}_1 \quad : \quad \theta > \theta_1.$$

We consider as alternatives a sequence of statistical models indexed by n and use a change of variable for the parameter $\theta = \theta_1 + \frac{u}{n}$ where $u \in U_n = [0, n(\beta - \theta_1)]$. Now the initial problem is reduced to the following one

$$\mathscr{H}_0 \quad : \quad u = 0,$$

$$\mathscr{H}_1 \quad : \quad u > 0.$$

Let us fixe $\varepsilon \in [0,1]$. Denote $\mathcal{K}_\varepsilon$ the class of tests functions $\overline{\Psi}_n(X^n)$ of asymptotic level ε i.e.

$$\mathcal{K}_\varepsilon = \left\{ \overline{\Psi}_n : \lim_{n\to\infty} \mathbf{E}_{\theta_1} \overline{\Psi}_n = \varepsilon \right\};$$

and the power function β_n of the test statistic is

$$\beta\left(\overline{\Psi}_n, u\right) = \mathbf{E}_{\theta_u}\left(\overline{\Psi}_n\right), \quad \theta_u = \theta_1 + \frac{u}{n}$$

where $\mathbf{E}_{\theta_1}$ and $\mathbf{E}_{\theta_u}$ are the mathematical expectation under hypothesis $\mathscr{H}_0$ and $\mathscr{H}_1$ respectively. Denote also by $\mathbf{P}_{\theta_1}^{(n)}$ and $\mathbf{P}_{\theta_u}^{(n)}$ the measure induced in the space of observations by n realizations of the Poisson process under hypothesis $\mathscr{H}_0$ and $\mathscr{H}_1$ respectively. The measures $\mathbf{P}_\theta^{(n)}$, $\theta \in \Theta$ are equivalent and the likelihood ratio function is

$$L\left(\theta, \theta_1, X^{(n)}\right) = \frac{L\left(\theta, X^{(n)}\right)}{L\left(\theta_1, X^{(n)}\right)} = \exp\left\{ \sum_{j=1}^n \int_0^\tau \ln\Big(\frac{\lambda_0 + \lambda_1 \mathbb{1}_{\{\theta \le t \le \theta + \tau_0\}}}{\lambda_0 + \lambda_1 \mathbb{1}_{\{\theta_1 \le t \le \theta_1 + \tau_0\}}} \Big) dX_j(t) \right.$$
$$\left. - n \int_0^\tau \lambda_1 \big(\mathbb{1}_{\{\theta \le t \le \theta + \tau_0\}} - \mathbb{1}_{\{\theta_1 \le t \le \theta_1 + \tau_0\}} \big) dt \right\}.$$

Note that the likelihood ratio $L\left(\theta,\theta_1,X^{(n)}\right)$ is a discontinuous function of θ but we have shown that the MLE $\hat\theta_n$ is the middle of the highest interval of the function $L\left(\cdot,\theta_1,X^{(n)}\right)$. Therefore it is defined as a solution of the following equation

$$L\left(\hat\theta_n,\theta_1,X^{(n)}\right)=\sup_{\theta\in[\theta_1,\beta)}L\left(\theta,\theta_1,X^{(n)}\right). \tag{3.24}$$

Further for $u,u_*>0$, introduce the four independent Poisson processes $X(\cdot)$, $Y(\cdot)$, $X^*(\cdot)$ and $Y^*(\cdot)$ such that

$$\mathbf{E}X(u)=(\lambda_0+\lambda_1)u,\quad \mathbf{E}Y(u)=\lambda_0 u$$

$$\mathbf{E}X^*(u_*)=\lambda_0 u_*\quad \mathbf{E}Y^*(u_*)=(\lambda_0+\lambda_1)u_*.$$

Define the random processes

$$Z(u)=\exp\left\{\rho\left(Y(u)-X(u)\right)\right\}$$

and

$$Z^*(u_*)=\exp\left\{\rho\left(Y^*(u_*)-X^*(u_*)\right)\right\}.$$

Then we put

$$\widetilde{Z}(u,u_*)=\frac{Z^*(u_*)}{Z^*(u)}\mathbb{1}_{\{u\leq u_*\}}+\frac{Z(u)}{Z(u_*)}\mathbb{1}_{\{u>u_*\}}.$$

3.6.2 The GLRT and Wald's Tests

Let

$$\hat G_n=L\left(\hat\theta_n,\theta_1,X^{(n)}\right)=\sup_{\theta\in[\theta_1,\beta)}L\left(\theta,\theta_1,X^{(n)}\right).$$

Then the Generalized Likelihood Ratio Test (GLRT) is given by the decision function

$$\overline{\Psi}_n=\mathbb{1}_{\{\hat G_n>c_\varepsilon\}}$$

and the Wald test is given by the decision function

$$\hat\Psi_n=\mathbb{1}_{\{n(\hat\theta_n-\theta_1)>b_\varepsilon\}}$$

where the thresholds c_ε and b_ε are chosen according to the condition $\overline{\Psi}_n\in\mathcal{K}_\varepsilon$ and $\hat\Psi_n\in\mathcal{K}_\varepsilon$ respectively. Thus the main results are given by the following theorems

Theorem 8 *Suppose that the value c_ε is a solution of the equation*

$$\mathbf{P}\{\sup_{u>0} Z(u) > c_\varepsilon\} = \varepsilon.$$

Then the GLRT test

$$\overline{\Psi}_n \in \mathcal{K}_\varepsilon$$

and its power function converges to the following limit

$$\beta\left(\overline{\Psi}_n, u_*\right) \longrightarrow \mathbf{P}\{\sup_{u>0} Z^*(u_*)\widetilde{Z}(u, u_*) > c_\varepsilon\}.$$

Theorem 9 *Suppose that the value b_ε is a solution of the equation*

$$\mathbf{P}\{\hat{u} > b_\varepsilon\} = \varepsilon.$$

Then the WT test

$$\hat{\Psi}_n \in \mathcal{K}_\varepsilon$$

and its power function converges to the following limit

$$\beta\left(\hat{\Psi}_n, u_*\right) \longrightarrow \mathbf{P}\{\hat{u}_{u_*} > b_\varepsilon\}.$$

where $\hat{u}$ and $\hat{u}_{u_*}$ are random variables such that

$$Z(\hat{u}) = \sup_{u\in\mathbb{R}^+} Z(u), \quad Z(\hat{u}_{u_*}, u_*) = \sup_{u\in\mathbb{R}^+} \widetilde{Z}(u, u_*)$$

respectively.

Introduce the normalized likelihood ratio

$$Z_n(u) = \frac{L\left(\theta_1 + \frac{u}{n}, X^{(n)}\right)}{L\left(\theta_1, X^{(n)}\right)}, \quad u \in (0, (\beta - \theta_1)n).$$

The considered tests are functionals of the likelihood ratio $L\left(\cdot, X^{(n)}\right)$. Therefore the GLRT and Wald tests can be written as functionals of the normalized likelihood ratio $Z_n(\cdot)$. Consequently we have to prove the weak convergence of the measures induced by the normalized likelihood ratio under hypothesis(to find the thresholds) and under alternative (to describe the power functions).

Let $\mathbf{D}_0(\mathbb{R}^+)$ be the space of function $\varphi(u)$ without discontinuities of the second kind defined on $\mathbb{R}^+$ and such that $\lim_{|u|\to+\infty} \varphi(u)$=0. We assume that all the function

$\varphi(u) \in \mathbf{D}_0(\mathbb{R}^+)$ are cadlàg on $\mathbb{R}^+$. Letφ_1 and φ_2 be two functions belonging to $\mathbf{D}_0(\mathbb{R}^+)$. The Skorohod distance between $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ is defined as follows

$$d(\varphi_1, \varphi_2) = \inf_{\mu} \left[\sup_{\mathbb{R}^+} |\varphi_1(u) - \varphi_2(\mu(u))| + \sup_{\mathbb{R}^+} |u - \mu(u)| \right],$$

where the lower bound is taken over all the increasing continuous one-to-one mappings $\mu : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. Let us also denote

$$\begin{aligned} \Delta_h(z) &= \sup_{u \in \mathbb{R}^+} \sup_{u-h \leq u' < u < u" \leq u+h} \left[\min \left\{ \left| z(u') - z(u) \right|, \left| z(u") - z(u) \right| \right\} \right] \\ &+ \sup_{|u| > h^{-1}} |z(u)| . \end{aligned}$$

This metric space $(\mathbf{D}_0(\mathbb{R}^+), d(\cdot, \cdot))$ is complete separable. Suppose that we have a sequence $(z_n)_{n \geq 1}$ of stochastic processes $z_n = \left\{ z_n(u), u \in \mathbb{R}^+ \right\}$ and a process $z = \left\{ z(u), u \in \mathbb{R}^+ \right\}$ such that realizations of these processes belong to the space $\mathbf{D}_0(\mathbb{R}^+)$. Denote $\mathbf{Q}^n$ and $\mathbf{Q}$ the distributions (which we suppose depending on a parameter $\theta \in \Theta$) induced on the measurable space $(\mathbf{D}_0(\mathbb{R}^+), \mathcal{B}(\mathbb{R}^+))$ by these processes. Here $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-algebra of the metric space $\mathbf{D}_0(\mathbb{R}^+)$. A criterion of weak convergence in $\mathbf{D}_0(\mathbb{R}^+)$ is given in the following lemma (see [32] for more details).

Lemma 14 *Let the following two conditions be satisfied:*

1- *the finite dimensional distributions of the process z_n converge to the finite dimensional distributions of the process z uniformly in $\theta \in \mathbf{K} \subset \Theta$.*

2- *For any $\delta > 0$, we have*

$$\lim_{h \to 0} \sup_{n \in \mathbb{N}} \sup_{\theta \in \mathbf{K}} \mathbf{Q}^n \left\{ \Delta_h(z_n) > \delta \right\} = 0. \tag{3.25}$$

Then for all functionals $\phi(\cdot) \in \mathbf{D}_0(\mathbb{R})$ the distribution of $\phi(z_n)$ converges to the distribution of $\phi(z)$ uniformly in $\theta \in \mathbf{K}$, that is, z_n converge weakly uniformly to z.

Furthermore we define the process $Z_n(u)$ as a linear decreasing to zero on the interval $(0, (\beta - \theta_1)n + 1)$ and put $Z_n(u) = 0$ for $u > (\beta - \theta_1)n + 1$. Now with probability 1 the realizations of $Z_n(\cdot)$ belongs to $\mathbf{D}_0(\mathbb{R}^+)$. Under the hypothesis i.e. $\mathscr{H}_0$, the Theorem 4 (Section 3.2) guarantee the weak convergence of $Z_n(\cdot)$ to the process $Z(\cdot)$ in $\mathbf{D}_0(\mathbb{R}^+)$.

3.6.3 Weak convergence under alternative

To gives the weak convergence of $Z_n(\cdot)$ under alternative we proceed as follows. Recall

$$Z_n(u) = \frac{L\left(\theta_1 + \frac{u}{n}, X^{(n)}\right)}{L\left(\theta_1, X^{(n)}\right)}, \quad u \in (0, (\beta - \theta_1)n).$$

For $u, u_* \in (0, (\beta - \theta_1)n + 1)$, we can write

$$\begin{aligned} Z_n(u) &= \frac{L\left(\theta_1 + \frac{u}{n}, X^{(n)}\right)}{L\left(\theta_1, X^{(n)}\right)} = \frac{L\left(\theta_1 + \frac{u}{n}, X^{(n)}\right)}{L\left(\theta_1, X^{(n)}\right)} \times \frac{L\left(\theta_1 + \frac{u_*}{n}, X^{(n)}\right)}{L\left(\theta_1 + \frac{u_*}{n}, X^{(n)}\right)} \\ &= \frac{L\left(\theta_1 + \frac{u}{n}, X^{(n)}\right)}{L\left(\theta_1 + \frac{u_*}{n}, X^{(n)}\right)} \times \frac{L\left(\theta_1 + \frac{u_*}{n}, X^{(n)}\right)}{L\left(\theta_1, X^{(n)}\right)} \\ &= \widetilde{Z}_n(u, u_*) Z_n(u_*). \end{aligned}$$

The process has the following representation

$$\ln \widetilde{Z}_n(u, u_*) := \begin{cases} \rho \sum_{j=1}^n \left\{ \int_{\theta_1 + \frac{u_*}{n} + \tau_0}^{\theta_1 + \frac{u}{n} + \tau_0} dX_j(t) - \int_{\theta_1 + \frac{u_*}{n}}^{\theta_1 + \frac{u}{n}} dX_j(t) \right\} & u \geq u_* \\ \rho \sum_{j=1}^n \left\{ \int_{\theta_1 + \frac{u_*}{n}}^{\theta_1 + \frac{u}{n}} dX_j(t) - \int_{\theta_1 + \frac{u_*}{n} + \tau_0}^{\theta_1 + \frac{u}{n} + \tau_0} dX_j(t) \right\} & u < u_* \end{cases}$$

where $\rho = \ln \frac{\lambda_0 + \lambda_1}{\lambda_0}$.

Define the process $\widetilde{Z}_n(u, u_*)$ as a linear decreasing to zero on the interval $(0, (\beta - \theta_1)n + 1)$ and put $Z_n(u, u_*) = 0$ for $u > (\beta - \theta_1)n + 1$. Now with probability 1 the realizations of $\widetilde{Z}_n(\cdot, u_*)$ belong to $\mathbf{D}_0(\mathbb{R}^+)$.

Denote $\mathbf{Q}^n_{u_*}$ and $\mathbf{Q}_{u_*}$ the distributions induced on the separable metric space $(\mathbf{D}_0(\mathbb{R}^+), \mathcal{B}(\mathbb{R}^+))$ by the processes $\{\widetilde{Z}_n(\cdot, u_*), \widetilde{Z}_n(u_*)\}$ and $\{\widetilde{Z}(\cdot, u_*), Z(u_*)\}$ respectively. Here $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-algebra of the metric space $\mathbf{D}_0(\mathbb{R}^+)$. The following proposition gives us the weak convergence in $\mathbf{D}_0(\mathbb{R}^+)$ under the alternative $\theta_{u_*} = \theta_1 + \frac{u_*}{n}$

Proposition 6 *Then under the alternative $\theta_{u_*} = \theta_1 + \frac{u_*}{n}$, we have the convergence*

$$\mathbf{Q}^n_{u_*} \Longrightarrow \mathbf{Q}_{u_*}$$

The proof is based on several lemmas. We verify the convergence of the finite dimensional distributions and the condition (3.25).

Lemma 15 *For fixed u_*, under the alternative hypothesis the finite dimensional distributions of the process $\{\widetilde{Z}_n(u, u_*), Z_n(u_*)\}$ converges to the finite dimensional distributions of the process $\{\widetilde{Z}(u, u_*), Z(u_*)\}$, $u \in \mathbb{R}^+$.*

Proof Let us calculate the characteristic function

$$\begin{aligned}\Phi_n(y_1,y) &= \mathbf{E}_{\theta_{u_*}} \exp[iy_1 \ln Z_n(u_*) + iy \ln \widetilde{Z}_n(u,u_*)] \\ &= I_n(y_1,y) + J_n(y_1,y)\end{aligned}$$

where

$$I_n(y_1,y) = \mathbf{E}_{\theta_{u_*}} \exp[iy_1 \ln Z_n(u_*) + iy \ln \widetilde{Z}_n(u,u_*)], \quad u_* > u$$

and

$$J_n(y_1,y) = \mathbf{E}_{\theta_{u_*}} \exp[iy_1 \ln Z_n(u_*) + iy \ln \widetilde{Z}_n(u,u_*)], \quad u_* \le u.$$

The Poisson processes $X(\cdot)$, $Y(\cdot)$, $X^*(\cdot)$ and $Y^*(\cdot)$ are independent. Therefore we have

$$J_n(y_1,y) = \mathbf{E}_{\theta_{u_*}} \exp[iy_1 \ln Z_n(u_*)] \mathbf{E}_{\theta_{u_*}} \exp[iy \ln \widetilde{Z}_n(u,u_*)].$$

$$\begin{aligned}&\ln \mathbf{E}_{\theta_{u_*}} \exp[iy_1 \ln Z_n(u_*)] \\ &= n\int_0^\tau \left[\exp\left(iy_1 \ln \frac{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_*}\}}}{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta\}}}\right) - 1\right]\left(\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_*}\}}\right) dt \\ &= n\int_{\theta_1}^{\theta_1+\frac{u_*}{n}} \left[\exp\left(iy_1 \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right) - 1\right]\lambda_0 dt \\ &\quad + n\int_{\theta_1+\tau_0}^{\theta_1+\tau_0+\frac{u_*}{n}} \left[\exp\left(iy_1 \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right) - 1\right][\lambda_0+\lambda_1]\, dt \\ &\longrightarrow u_*\lambda_0 \left[\exp\left(iy_1 \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right) - 1\right] \\ &\quad + u_*[\lambda_0+\lambda_1]\left[\exp\left(iy_1 \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right) - 1\right] = \Phi(y_1).\end{aligned}$$

Further

$$\begin{aligned}&\mathbf{E}\exp\{i\, y \ln Z^*(u_*)\} \\ &= \mathbf{E}\exp\left\{i\, y_1 \ln\left(\frac{\lambda_0}{\lambda_0+\lambda_1}\right) X^*(u_*) + iy_1 \ln\left(\frac{\lambda_0+\lambda_1}{\lambda_0}\right) Y^*(u_*)\right\} \\ &= \mathbf{E}\exp\left\{iy_1 \ln\left(\frac{\lambda_0}{\lambda_0+\lambda_1}\right) X^*(u_*)\right\}\mathbf{E}\exp\left\{iy_1 \ln\left(\frac{\lambda_0+\lambda_1}{\lambda_0}\right) Y^*(u_*)\right\}.\end{aligned}$$

The Poisson processes $X^*(\cdot)$, $Y^*(\cdot)$ are independent, then we obtain the following calculation

$$
\begin{aligned}
&\mathbf{E}\exp\left\{iy_1\ln\left(\frac{\lambda_0}{\lambda_0+\lambda_1}\right)X^*(u_*)\right\}\\
&\qquad=\exp\left\{\int_0^{u_*}\left[\exp\left(iy_1\ln\frac{\lambda_0}{\lambda_0+\lambda_1}\right)-1\right]\lambda_0 dt\right\}\\
&\qquad=\exp\left\{u_*\left[\exp\left(iy_1\ln\frac{\lambda_0}{\lambda_0+\lambda_1}\right)-1\right]\lambda_0\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{E}\exp\left\{iy_1\ln\left(\frac{\lambda_0+\lambda_1}{\lambda_0}\right)Y^*(u_*)\right\}\\
&\qquad=\exp\left\{\int_0^{u_*}\left[\exp\left(iy_1\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\right)-1\right](\lambda_0+\lambda_1) dt\right\}\\
&\qquad=\exp\left\{u_*\left[\exp\left(iy_1\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\right)-1\right](\lambda_0+\lambda_1)\right\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbf{E}\exp\left\{i\,y\ln Z^*(u_*)\right\}&=\exp\left\{u_*\left[\exp\left(iy_1\ln\frac{\lambda_0}{\lambda_0+\lambda_1}\right)-1\right]\lambda_0\right\}\\
&\qquad\times\exp\left\{u_*\left[\exp\left(iy_1\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\right)-1\right](\lambda_0+\lambda_1)\right\}\\
&=\exp\left\{u_*\left[\exp\left(iy_1\ln\frac{\lambda_0}{\lambda_0+\lambda_1}\right)-1\right]\lambda_0\right.\\
&\qquad\left.+u_*\left[\exp\left(iy_1\ln\frac{\lambda_0+\lambda_1}{\lambda_0}\right)-1\right](\lambda_0+\lambda_1)\right\}=\Phi(y_1).
\end{aligned}
$$

Therefore

$$
\ln\mathbf{E}_{\theta_{u_*}}\exp[iy_1\ln Z_n(u_*)]\longrightarrow\ln\mathbf{E}\exp[iy_1\ln Z^*(u_*)]
$$

We also have

$$
\begin{aligned}
&\ln \mathbf{E}_{\theta_u} \exp[iy_1 \ln \widetilde{Z}_n(u,u_*)] \\
&\quad = n\int_0^{\tau}\left[\exp\left(iy_1 \ln \frac{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_u\}}}{\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_*}\}}}\right)-1\right]\left(\lambda_0+\lambda_1 1\!\!1_{\{t\in\Delta_{u_*}\}}\right)dt \\
&\quad = n\int_{\theta_1+\frac{u_*}{n}}^{\theta_1+\frac{u}{n}}\left[\exp\left(iy_1 \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right)-1\right](\lambda_0+\lambda_1)dt \\
&\qquad + n\int_{\theta_1+\tau_0+\frac{u_*}{n}}^{\theta_1+\tau_0+\frac{u}{n}}\left[\exp\left(iy_1 \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right)-1\right]\lambda_0 dt \\
&\quad \longrightarrow (u-u_*)(\lambda_0+\lambda_1)\left[\exp\left(iy_1 \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right)-1\right] \\
&\qquad + (u-u_*)\lambda_0\left[\exp\left(iy_1 \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right)-1\right].
\end{aligned}
$$

Further

$$
\ln \frac{Z(u)}{Z(u_*)} = \ln \frac{\lambda_0}{\lambda_0+\lambda_1} X(u-u_*) + \ln \frac{\lambda_0+\lambda_1}{\lambda_0} Y(u-u_*).
$$

$$
\begin{aligned}
\ln \mathbf{E} \exp[iy_1 \ln \frac{Z(u)}{Z(u_*)}] &= \int_0^{u-u_*}\left[\exp\left(iy_1 \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right)-1\right](\lambda_0+\lambda_1)dt \\
&\quad + \int_0^{u-u_*}\left[\exp\left(iy_1 \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right)-1\right]\lambda_0 dt \\
&= (u-u_*)(\lambda_0+\lambda_1)\left[\exp\left(iy_1 \ln \frac{\lambda_0}{\lambda_0+\lambda_1}\right)-1\right] \\
&\quad + (u-u_*)\lambda_0\left[\exp\left(iy_1 \ln \frac{\lambda_0+\lambda_1}{\lambda_0}\right)-1\right]
\end{aligned}
$$

which provides the convergence

$$
\ln \mathbf{E}_{\theta_u} \exp[iy_1 \ln \widetilde{Z}_n(u,u_*)] \longrightarrow \ln \mathbf{E} \exp[iy_1 \ln \frac{Z(u)}{Z(u_*)}] = \ln \mathbf{E} \exp[iy_1 \ln \widetilde{Z}(u,u_*)].
$$

In a similar way we give the proof for $u < u_*$. Consequently

$$
I_n(y_1,y_2) \longrightarrow \mathbf{E} \exp\{i\, y \ln Z^*(u_*)\}\, \mathbf{E} \exp[iy_1 \ln \widetilde{Z}(u,u_*)] \qquad \square
$$

To end the proof of the Proposition 6, the other lemma can be verified for the process $\widetilde{Z}_n(u,u_*)$, for fixed u_*.

Lemma 16 *For $u_*, u_1, u_2 \in (0, n(\beta - \theta_1))$ there exist constants $C_1, C_2, C_3, C_4 > 0$ such that*

$$\mathbf{E}_{\theta_{u_*}} \mid \widetilde{Z}_n^{1/2}(u_1, u_*) - \widetilde{Z}_n^{1/2}(u_1, u_*) \mid^2 \leq C_1 \mid u_1 - u_2 \mid, \tag{3.26}$$

$$\mathbf{E}_{\theta_{u_*}} \widetilde{Z}_n^{1/2}(u, u_*) \leq e^{-C_2|u-u_*|}. \tag{3.27}$$

Proof. Let

$$\widetilde{Z}_n(u, u_*) = \frac{d\mathbf{P}_{\theta_1 + \frac{u}{n}}}{d\mathbf{P}_{\theta_1 + \frac{u_*}{n}}} = \exp\Big\{\sum_{j=1}^{n} \int_0^\tau \ln \frac{\lambda(\theta_1 + \frac{u}{n}, t)}{\lambda(\theta_1 + \frac{u_*}{n}, t)} dX_j(t)\Big\}.$$

According to the proposition 5 (Chapter 1), we have

$$\begin{aligned}
&\mathbf{E}_{\theta_{u_*}} \mid \widetilde{Z}_n^{1/2}(u_1, u_*) - \widetilde{Z}_n^{1/2}(u_1, u_*) \mid^2 \\
&\quad \leq n \int_0^\tau \left(\sqrt{\frac{\lambda(\theta_1 + \frac{u_1}{n}, t)}{\lambda(\theta_1 + \frac{u_*}{n}, t)}} - \sqrt{\frac{\lambda(\theta_1 + \frac{u_2}{n}, t)}{\lambda(\theta_1 + \frac{u_*}{n}, t)}}\right)^2 \lambda(\theta_1 + \frac{u_*}{n}, t) dt.
\end{aligned}$$

There exist several situations which depend on the position between u_2, u_1, u_* and τ_0. We treat some of them and the others cases can be obtained in a similar way.

First we suppose that $0 < u_2 < u_1 < u_*$ and $\tau_0 > \frac{u_*}{n}$. Then

$$\begin{aligned}
&n \int_0^\tau \left(\sqrt{\frac{\lambda(\theta_1 + \frac{u_1}{n}, t)}{\lambda(\theta_1 + \frac{u_*}{n}, t)}} - \sqrt{\frac{\lambda(\theta_1 + \frac{u_2}{n}, t)}{\lambda(\theta_1 + \frac{u_*}{n}, t)}}\right)^2 \lambda(\theta_1 + \frac{u_*}{n}, t) dt \\
&= n \int_{\theta_1 + \frac{u_2}{n}}^{\theta_1 + \frac{u_1}{n}} \left(1 - \sqrt{\frac{\lambda_0 + \lambda_1}{\lambda_0}}\right)^2 \lambda_0 dt + n \int_{\theta_1 + \frac{u_2}{n} + \tau_0}^{\theta_1 + \frac{u_1}{n} + \tau_0} \left(1 - \sqrt{\frac{\lambda_0}{\lambda_0 + \lambda_1}}\right)^2 (\lambda_0 + \lambda_1) dt \\
&= C_1(u_1 - u_2) + C_2(u_1 - u_2) = C(u_1 - u_2).
\end{aligned}$$

Further if $0 < u_2 < u_* < u_1$ and $\tau_0 > \frac{u_1}{n}$. Then

$$\begin{aligned}
&n\int_0^{\tau}\left(\sqrt{\frac{\lambda(\theta_1+\frac{u_1}{n},t)}{\lambda(\theta_1+\frac{u_*}{n},t)}}-\sqrt{\frac{\lambda(\theta_1+\frac{u_2}{n},t)}{\lambda(\theta_1+\frac{u_*}{n},t)}}\right)^2\lambda(\theta_1+\frac{u_*}{n},t)dt\\
&=n\int_{\theta_1+\frac{u_2}{n}}^{\theta_1+\frac{u_*}{n}}\left(1-\sqrt{\frac{\lambda_0+\lambda_1}{\lambda_0}}\right)^2\lambda_0dt+n\int_{\theta_1+\frac{u_*}{n}}^{\theta_1+\frac{u_1}{n}}\left(\sqrt{\frac{\lambda_0}{\lambda_0+\lambda_1}}-1\right)^2(\lambda_0+\lambda_1)dt\\
&+n\int_{\theta_1+\frac{u_2}{n}+\tau_0}^{\theta_1+\frac{u_*}{n}+\tau_0}\left(1-\sqrt{\frac{\lambda_0}{\lambda_0+\lambda_1}}\right)^2(\lambda_0+\lambda_1)dt+n\int_{\theta_1+\frac{u_*}{n}+\tau_0}^{\theta_1+\frac{u_1}{n}+\tau_0}\left(\sqrt{\frac{\lambda_0+\lambda_1}{\lambda_0}}-1\right)^2\lambda_0dt\\
&=\left(1-\sqrt{\frac{\lambda_0+\lambda_1}{\lambda_0}}\right)^2\lambda_0(u_*-u_2)+\left(1-\sqrt{\frac{\lambda_0+\lambda_1}{\lambda_0}}\right)^2(\lambda_0+\lambda_1)(u_1-u_*)\\
&\quad+\left(1-\sqrt{\frac{\lambda_0}{\lambda_0+\lambda_1}}\right)^2(\lambda_0+\lambda_1)(u_*-u_2)+\left(\sqrt{\frac{\lambda_0+\lambda_1}{\lambda_0}}-1\right)^2\lambda_0(u_1-u_*)\\
&=\left(1-\sqrt{\frac{\lambda_0+\lambda_1}{\lambda_0}}\right)^2\lambda_0(u_*-u_2+u_1-u_*)\\
&\quad+\left(1-\sqrt{\frac{\lambda_0}{\lambda_0+\lambda_1}}\right)^2(\lambda_0+\lambda_1)(u_1-u_*+u_*-u_2)\\
&=\left(1-\sqrt{\frac{\lambda_0+\lambda_1}{\lambda_0}}\right)^2\lambda_0(u_1-u_2)+\left(1-\sqrt{\frac{\lambda_0}{\lambda_0+\lambda_1}}\right)^2(\lambda_0+\lambda_1)(u_1-u_2)\\
&=C(u_1-u_2).
\end{aligned}$$

For $0 < u_* < u_2 < u_1$ and $\tau_0 > \frac{u_1}{n}$, we have

$$\begin{aligned}
&n\int_0^{\tau}\left(\sqrt{\frac{\lambda(\theta_1+\frac{u_1}{n},t)}{\lambda(\theta_1+\frac{u_*}{n},t)}}-\sqrt{\frac{\lambda(\theta_1+\frac{u_2}{n},t)}{\lambda(\theta_1+\frac{u_*}{n},t)}}\right)^2\lambda(\theta_1+\frac{u_*}{n},t)dt\\
&=n\int_{\theta_1+\frac{u_2}{n}}^{\theta_1+\frac{u_1}{n}}\left(\sqrt{\frac{\lambda_0}{\lambda_0+\lambda_1}}-1\right)^2(\lambda_0+\lambda_1)dt\\
&\quad+n\int_{\theta_1+\frac{u_2}{n}+\tau_0}^{\theta_1+\frac{u_1}{n}+\tau_0}\left(\sqrt{\frac{\lambda_0+\lambda_1}{\lambda_0}}-1\right)^2\lambda_0dt\\
&=\left(\sqrt{\frac{\lambda_0}{\lambda_0+\lambda_1}}-1\right)^2\lambda_0(u_1-u_2)+\left(\sqrt{\frac{\lambda_0+\lambda_1}{\lambda_0}}-1\right)^2\lambda_0(u_1-u_2)\\
&=C(u_1-u_2).
\end{aligned}$$

Hence the inequality (3.26) is proved.

Let $0 < u < u_*$ and $\tau_0 > \frac{u_*}{n}$(the others cases can be shown by a similar way).

Then according to the proposition 5 (Chapter 1), we have

$$\mathbf{E}_{\theta_{u_*}} \widetilde{Z}_n^{1/2}(u,u_*) = \exp\biggl\{ \frac{n}{2} \int_0^\tau \left(\frac{\lambda(\theta_1 + \frac{u}{n}, t)}{\lambda(\theta_1 + \frac{u_*}{n}, t)} - 1 \right)^2 \lambda(\theta_1 + \frac{u_*}{n}, t) dt \biggr\}$$
$$= \exp\biggl\{ -\frac{n}{2} \int_{\theta_1+\frac{u}{n}}^{\theta_1+\frac{u_*}{n}} \left(\frac{\lambda_0+\lambda_1}{\lambda_0} - 1 \right)^2 \lambda_0 dt$$
$$- \frac{n}{2} \int_{\theta_1+\frac{u}{n}+\tau_0}^{\theta_1+\frac{u_*}{n}+\tau_0} \left(\frac{\lambda_0}{\lambda_0+\lambda_1} - 1 \right)^2 (\lambda_0+\lambda_1) dt \biggr\}$$
$$= \exp\biggl\{ -\frac{1}{2} \left(\frac{\lambda_0+\lambda_1}{\lambda_0} - 1 \right)^2 \lambda_0 (u_* - u) - \frac{1}{2} \left(\frac{\lambda_0}{\lambda_0+\lambda_1} - 1 \right)^2 (\lambda_0+\lambda_1)(u_* - u) \biggr\}$$
$$= \exp\biggl\{ -\frac{1}{2} \left[\left(\frac{\lambda_0+\lambda_1}{\lambda_0} - 1 \right)^2 \lambda_0 + \left(\frac{\lambda_0}{\lambda_0+\lambda_1} - 1 \right)^2 (\lambda_0+\lambda_1) \right] (u_* - u) \biggr\}$$
$$= \exp\bigl\{ -k(u_* - u) \bigr\}.$$

The inequality (3.27) is proved.

Lemma 17 *There exist constants* $C_3, C_4 > 0$ *such that*

$$\mathbf{P}_{\theta_{u_*}}^{(n)}(\mathbb{D}^*{}_1) \leq C_3 h, \quad \mathbf{P}_{\theta_{u_*}}^{(n)}(\mathbb{D}^*{}_2) \leq C_4^2 h^2.$$

where $\mathbb{D}^*$ is the event that on the interval $[l, l+1]$ there exist at least two jumps of the process $\widetilde{Z}_n(u,u_*)$ such that the distance between them is less than $2h$. Denote $\mathbb{D}^*{}_p$ the event that the process $\widetilde{Z}_n(u,u_*)$ has at least p jumps on the interval $(u, u+h)$ and $(u+\tau_0, u+\tau_0+h)$.

The Lemmas 15, 16 and 17 (see the Theorem 5.2 [41]) gives the weak convergence (under the alternative for $\theta_{u_*} = \theta_1 + \frac{u_*}{n}$) of the process $\{\widetilde{Z}_n(u,u_*), Z_n(u_*)\}$ to the process $\{\widetilde{Z}(u,u_*), Z(u_*)\}$,$u \in \mathbb{R}^+$.

Proof of Theorem 8: Under the hypothesis we have

$$\mathbf{P}_{\theta_1}^{(n)}\{\hat{G}_n > c_\varepsilon\} = \mathbf{P}_{\theta_1}^{(n)}\{\sup_{u \in U_n} Z_n(u) > c_\varepsilon\} \longrightarrow \mathbf{P}\{\sup_{u>0} Z(u) > c_\varepsilon\}.$$

The threshold c_ε is the solution of the equation

$$\mathbf{P}\{\sup_{u>0} Z(u) > c_\varepsilon\} = \varepsilon.$$

Under the alternative $\theta_{u_*} = \theta_1 + \frac{u_*}{n}$ ($u_* \in U_n$ fixed). In view of the weak convergence we obtain

$$\begin{aligned} \beta\left(\overline{\Psi}_n, u_*\right) &= \mathbf{E}_{\theta_u}\left(\overline{\Psi}_n\right) \\ &= \mathbf{P}_{\theta_{u_*}}^{(n)}\bigl\{\sup_{u>0} Z_n(u) > c_\varepsilon\bigr\} \longrightarrow \mathbf{P}\{\sup_{u>0} Z^*(u_*)\widetilde{Z}(u,u_*) > c_\varepsilon\}. \qquad \square \end{aligned}$$

Proof of Theorem 9: Under the hypothesis we have the convergence in distribution (see the part of parametric estimation) $n(\hat{\theta}_n - \theta_1) \Rightarrow \hat{u}$. Thus the threshold is given as the solution of the equation

$$\mathbf{P}_0\{\hat{u} > b_\varepsilon\} = \varepsilon.$$

Under the alternative $\theta_{u_*} = \theta_1 + \frac{u_*}{n}$ ($u_* \in U_n$ fixed) we have

$$\begin{aligned}
\beta\left(\overline{\Psi}_n, u_*\right) &= \mathbf{E}_{\theta_{u_*}}\left(\overline{\Psi}_n\right) \\
&= \mathbf{P}^{(n)}_{\theta_{u_*}}\left\{n(\hat{\theta}_n - \theta_1) > c_\varepsilon\right\} \\
&= \mathbf{P}^{(n)}_{\theta_{u_*}}\left\{\sup_{n(\theta-\theta_1)>c_\varepsilon} L(\theta, X^n) > \sup_{n(\theta-\theta_1)\le c_\varepsilon} L(\theta, X^n)\right\} \\
&= \mathbf{P}^{(n)}_{\theta_{u_*}}\left\{\sup_{n(\theta-\theta_1)>c_\varepsilon} \frac{L(\theta, X^n)}{L(\theta_1, X^n)} > \sup_{n(\theta-\theta_1)\le c_\varepsilon} \frac{L(\theta, X^n)}{L(\theta_1, X^n)}\right\} \\
&= \mathbf{P}^{(n)}_{\theta_{u_*}}\left\{\sup_{u>c_\varepsilon} Z_n(u) > \sup_{u\le c_\varepsilon} Z_n(u)\right\} \\
&= \mathbf{P}^{(n)}_{\theta_{u_*}}\left\{\sup_{u>c_\varepsilon} \widetilde{Z}_n(u, u_*) > \sup_{u\le c_\varepsilon} \widetilde{Z}_n(u, u_*)\right\} \\
&\longrightarrow \mathbf{P}\left\{\sup_{u>c_\varepsilon} \widetilde{Z}(u, u_*) > \sup_{u\le c_\varepsilon} \widetilde{Z}(u, u_*)\right\} = \mathbf{P}\{\hat{u}_{u_*} > b_\varepsilon\}. \qquad \square
\end{aligned}$$

Chapter 4

On the Cramér-von Mises test for Poisson processes with scale parameter

4.1 Introduction

The Poisson process is one of the simplest stochastic processes and that is why it is often considered as the first mathematical model in many applications. Therefore the problem of goodness of fit test for this model is important and our work is devoted to this problem. In this work we consider the problem of goodness of fit test for inhomogeneous Poisson process with composite parametric basic hypothesis with shift parameter and a weighted integral statistic.

The general theory of the goodness of fit tests in classical statistics is now well developed, see, e.g. [22], [44], [24], [20], [46] and the references therein. There is a large amount of literature on the applications of Poisson process models in different domains (astronomy, biology, image analysis, medicine, optical communication, physics, reliability theory, etc.). At the same time, the identification of many important models of Poisson processes (as well as a general theory of estimation) has not yet been well developed, and such an attempt would help to cover this gap. We also note that the class of inhomogeneous Poisson processes is quite rich and is an interesting model for statistical investigation.

Further it is known that in the case of a simple basic hypothesis, the tests of Cramér-von Mises, Kolmogorov-Smirnov, and some others are asymptotically distribution free. Therefore the choice of the threshold for these tests can be easily done. The situation is different if the basic hypothesis is composite parametric. Then in the general case, there is no distribution free property and the limit distributions of the corresponding statistics depend on the distribution function of the observed random variables and depends as well on the value of an unknown parameter. There are two particular cases when the limit distribution does not depend on the true value: the first one is the case of shift parameter and the second is the case of the scale parameter (see, e.g., [54], [46]). In these two cases the choice of the threshold can be done directly before the experiment. Another possibility to have asymptotically distribution free test occurs in the case of singular estimation, when the rate of convergence of the estimator is better than $\sqrt{n}$ [21]. Due to this rate, the asymptotic distribution of the test statistics does not depend on the underlying model.

The problems of goodness of fit test for Poisson processes were considered by many authors. Let us mention here the works [23], [17], [18], [13], [11]. In the paper [11] it was shown that for the model of inhomogeneous Poisson processes with parametric basic hypotheses in the situation when the unknown parameter is a shift parameter, the limit distribution of the Cramér-von Mises statistics under hypothesis does not depend on the unknown parameter.

In this work we study the goodness of fit test with parametric basic hypotheses in the case where the unknown parameter is a scale parameter. We show that the limit distribution of the Cramér-von Mises type statistic under the null hypothesis does not depend on the unknown parameter i.e. asymptotically parameter free (APF).

Suppose that we observe $X^{(n)} = (X_1, \ldots, X_n)$, n independent Poisson processes, where $X_j = (X_j(t), t \in \mathbb{R})$ are trajectories of the Poisson process with mean function $\Lambda(t) = \mathbf{E}X_j(t)$. Recall that if the basic (nulle) hypothesis is simple, say,

$$\mathscr{H}_0 \qquad : \qquad \Lambda(\cdot) = \Lambda_0(\cdot),$$

where $\Lambda_0(\cdot)$ is a known function with $\Lambda_0(\infty) < \infty$ and alternative

$$\mathscr{H}_1 \qquad : \qquad \Lambda(\cdot) \neq \Lambda_0(\cdot),$$

then we can introduce the Cramér-von Mises type statistic

$$\widetilde{\Delta}_n = \frac{n}{\Lambda_0(\infty)^2} \int_{-\infty}^{+\infty} \left[\widehat{\Lambda}_n(t) - \Lambda_0(t)\right]^2 \, d\Lambda_0(t).$$

Here

$$\widehat{\Lambda}_n(t) = \frac{1}{n} \sum_{j=1}^{n} X_j(t)$$

is the empirical mean of the Poisson process. It can be verified that this statistic converges to the following limit

$$\widetilde{\Delta}_n \Longrightarrow \Delta \equiv \int_0^1 W\left(s\right)^2 ds.$$

Here $W\left(s\right), 0 \leq s \leq 1$ is a standard Wiener process.

Therefore the test

$$\widetilde{\psi}_n\left(X^n\right) = \mathbb{1}_{\left\{\widetilde{\Delta}_n > c_\varepsilon\right\}}, \qquad \mathbf{P}\left\{\Delta > c_\varepsilon\right\} = \varepsilon,$$

is asymptotically distribution free (see, e.g., Dachian and Kutoyants [18]). Here $\varepsilon \in (0,\ 1)$ is the size of the test.

We are interested by the same problem but with the parametric basic hypothesis, i.e., we suppose that under hypothesis $\mathscr{H}_0$ the mean function belongs to a parametric family of functions and we propose a test of Cramér-von Mises type which is "asymptotically parameter free". This means that the limit distribution of the statistic does not depend on the unknown parameter. This result allows us to construct a test with asymptotically chosen probability of errors (under hypothesis). This test is consistent against any fixed alternative.

The similar problems for the ergodic diffusion processes were studied in [48], [42].

4.2 Asymptotically parameter free test

Let us introduce the intensity function

$$\Lambda_0\left(t,\theta\right) = \theta \int_{-\infty}^t \lambda_0\left(\frac{s}{\theta}\right) \frac{ds}{\theta} = \theta \Lambda_0\left(\frac{t}{\theta}\right)$$

and a parametric family

$$\mathcal{L}\left(\Theta\right) = \left\{\Lambda_0\left(t,\theta\right) = \theta\Lambda_0\left(\frac{t}{\theta}\right), \theta \in \Theta = \left(\alpha,\ \beta\right)\right\}$$

where $\Lambda_0\left(\cdot\right)$ is some known nondecreasing function with properties:

$$\Lambda_0\left(-\infty\right) = 0, \quad \Lambda_0\left(\infty\right) < \infty, \quad \Lambda_0\left(t\right) = \int_{-\infty}^t \lambda_0\left(s\right)\, ds.$$

We observe $X^{(n)} = (X_1, \ldots, X_n)$, n independent inhomogeneous Poisson processes, $X_j = \left\{X_j(t),\ t \in \mathbb{R}\right\}$ with the same mean function $\Lambda(\cdot)$. We replace the parameter θ by its maximum likelihood estimator $\hat{\theta}_n$ and our statistic is defined as follow

$$\Delta_n = \frac{n}{\hat{\theta}_n^2} \int_{-\infty}^{+\infty} \left[\widehat{\Lambda}_n(t) - \Lambda_0\left(t, \hat{\theta}_n\right)\right]^2 \lambda_0\left(\frac{t}{\hat{\theta}_n}\right)\, dt = \frac{\widetilde{\Delta}_n}{\hat{\theta}_n^2}$$

Therefore the Cramér-von Mises type test is

$$\hat{\Psi}_n\left(X^n\right) = \mathbb{I}_{\{\Delta_n > c_\varepsilon\}}.$$

The threshold c_ε must be chosen so that this test belongs to the class of tests of asymptotic level ε

$$\mathcal{K}_\varepsilon = \left\{\overline{\Psi}_n : \lim_{n\to\infty} \mathbf{E}_\theta \overline{\Psi}_n = \varepsilon, \quad \theta \in \Theta\right\}.$$

As we use the MLE $\hat{\theta}_n$, we need the following regularity conditions. Suppose that the intensity function $\lambda_0(\cdot)$ is strictly positive and sufficiently smooth. Under these conditions the MLE is consistent, asymptotically normal and the polynomial of moments converge (see [41]).

We have to test a composite parametric hypothesis

$$\mathscr{H}_0 \quad : \qquad \Lambda(\cdot) \in \mathcal{L}(\Theta)$$

against an alternative

$$\mathscr{H}_1 \quad : \qquad \Lambda(\cdot) \notin \mathcal{L}(\Theta).$$

More precisely, we suppose that under this alternative

$$\inf_{\theta\in\Theta} \left\|\Lambda(\cdot) - \Lambda_0(\cdot, \theta)\right\| > 0.$$

Here and the rest of the work, the notation $\|\cdot\|$ is the following L_2-norm.

$$\|f\|_\theta^2 = \int_{-\infty}^{\infty} f(t)^2 \lambda_0\left(\frac{t}{\theta}\right) dt.$$

We show that for such alternatives the test is consistent. This test will be uniformly consistent against another class of alternatives

$$\mathscr{H}_1^\rho \quad : \quad \Lambda(\cdot) \in \mathcal{F}_\rho = \left\{\Lambda(\cdot) : \inf_{\theta\in\Theta} \left\|\Lambda(\cdot) - \Lambda_0(\cdot, \theta)\right\|_\theta > \rho\right\}.$$

Here $\rho > 0$ is some given number. We suppose as well that $\mathcal{F}_\rho$ is such that

$$\sup_{\Lambda \in \mathcal{F}_\rho} \Lambda(\infty) < \infty.$$

The symbol dot stands for differentiation with respect to θ.

Let us introduce the following random variable

$$\Delta_0 = \int_{-\infty}^{\infty} \left[W(\Lambda_0(t)) + (\Lambda_0(t) - t\lambda_0(t)) \int_{-\infty}^{\infty} \mathrm{I}_0{}^{-1} s \frac{\dot{\lambda}_0(s)}{\lambda_0(s)} dW(\Lambda_0(s)) \right]^2 d\Lambda_0(t)$$

where $W(\cdot)$ is a standard Wiener process.

The constant c_ε is the solution of the equation

$$\mathbf{P}\{\Delta_0 > c_\varepsilon\} = \varepsilon.$$

Conditions A.

- **a1.** *The function* $\sqrt{\lambda_0(\cdot)} \in \mathcal{L}_2(\mathbb{R})$ *is strictly positive and three times continuously differentiable.*
- **a2.** *Its derivatives belong to* $\mathcal{L}_2(\mathbb{R})$ *.*
  The Fisher information

$$\mathrm{I}(\theta) = \frac{1}{\theta} \int_{-\infty}^{+\infty} t^2 \frac{\dot{\lambda}_0^2(t)}{\lambda_0(t)} dt > 0$$

- **a3.** *The derivative* $\dot{\lambda}_0(\cdot) \in \mathcal{L}_1(\mathbb{R})$ *.*

Note that under conditions **a1**- **a2** the MLE $\hat{\theta}_n$ is consistent, asymptotically normal

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}(\theta)^{-1}\right)$$

and the moments converge, i.e. for any $p > 0$

$$\lim_{n \to +\infty} \mathbf{E}_\theta \left| \sqrt{n}\left(\hat{\theta}_n - \theta\right) \right|^p = \mathrm{I}(\theta)^{-p/2} \mathbf{E}\left|\zeta\right|^p, \quad \zeta \sim \mathcal{N}(0,1).$$

Moreover, it admits the representation

$$\hat{\theta}_n = \theta + \varphi_n \left(\varphi_n \int_{-\infty}^{+\infty} \dot{l}(t,\theta)\, \pi^{(n)}(dt) \right) + O\left(\varphi_n \xi_n^{1/2}\right) \tag{4.1}$$

where

$$l(t,\theta) = \ln \lambda_0 \left(\frac{t}{\theta}\right), \quad \dot{l}(t,\theta) = -\frac{t}{\theta^2}\frac{\dot{\lambda}_0\left(\frac{t}{\theta}\right)}{\lambda_0\left(\frac{t}{\theta}\right)}$$

and

$$\pi^{(n)}(dt) = \sum_{j=1}^{n}\left[X_j(dt) - \lambda_0\left(\frac{t}{\theta}\right)dt\right].$$

Note that the Fisher Information is calculate as follows

$$I_n(\theta) = n\int_{-\infty}^{+\infty}\frac{\left[\frac{\partial\lambda_0}{\partial\theta}\left(\frac{s}{\theta}\right)\right]^2}{\lambda_0\left(\frac{s}{\theta}\right)}\,ds = n\int_{-\infty}^{+\infty}\frac{\frac{s^2}{\theta^4}\dot{\lambda}_0^2\left(\frac{s}{\theta}\right)}{\lambda_0\left(\frac{s}{\theta}\right)}\,ds.$$

If we put $\frac{s}{\theta}$= t, it becomes

$$I_n(\theta) = n\int_{-\infty}^{+\infty}\frac{t^2}{\theta^2}\frac{\dot{\lambda}_0^2(t)}{\lambda_0(t)}\theta\,dt = n\mathrm{I}(\theta) = \frac{n}{\theta^2}\mathrm{I}_0$$

where

$$\mathrm{I}(\theta) = \frac{1}{\theta^2}\int_{-\infty}^{+\infty}t^2\frac{\dot{\lambda}_0^2(t)}{\lambda_0(t)}\,dt$$

and

$$\mathrm{I}_0 = \int_{-\infty}^{+\infty}t^2\frac{\dot{\lambda}_0^2(t)}{\lambda_0(t)}\,dt.$$

Let

$$\varphi_n = \left(\frac{1}{nI\theta}\right)^{1/2} \quad \text{and} \quad \xi_n = n^{-1/2}.$$

Therefore from (4.1) we obtain

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) = -\frac{1}{\theta}I_0^{-1}\int_{-\infty}^{\infty}t\frac{\dot{\lambda}_0\left(\frac{t}{\theta}\right)}{\lambda_0\left(\frac{t}{\theta}\right)}\,dW_n(t) + O\left(n^{-1/4}\right). \tag{4.2}$$

Our main result is the following theorem.

Theorem 10 *Let the conditions* **A** *be fulfilled. Then the test*

$$\hat{\Psi}_n = 1\!\!1_{\{\Delta_n > c_\varepsilon\}} \in \mathcal{K}_\varepsilon.$$

Proof. We have the following equalities

$$\begin{aligned} u_n(t) &= \sqrt{n}\left(\widehat{\Lambda}_n(t) - \Lambda_0\left(t,\hat{\theta}_n\right)\right) \\ &= \sqrt{n}\left(\widehat{\Lambda}_n(t) - \Lambda_0\left(t,\theta_0\right) + \Lambda_0\left(t,\theta_0\right) - \Lambda_0\left(t,\hat{\theta}_n\right)\right) \\ &= \sqrt{n}\left(\widehat{\Lambda}_n(t) - \Lambda_0\left(t,\theta_0\right)\right) - \sqrt{n}\left(\Lambda_0\left(t,\hat{\theta}_n\right) - \Lambda_0\left(t,\theta_0\right)\right) \\ &= W_n(t) - \sqrt{n}\left(\Lambda_0\left(t,\hat{\theta}_n\right) - \Lambda_0\left(t,\theta_0\right)\right). \end{aligned} \tag{4.3}$$

Since the function $\Lambda_0\left(t,\theta\right)$ is differentiable on Θ, according to the formula of finite increments applied to Λ_0 on $\left[\theta_0,\, \hat{\theta}_n\right]$, we have

$$\Lambda_0\left(t,\hat{\theta}_n\right) - \Lambda_0\left(t,\theta_0\right) = \dot{\Lambda}_0\left(t,\widetilde{\theta}_n\right)\cdot\left(\hat{\theta}_n - \theta_0\right) \tag{4.4}$$

where $\widetilde{\theta}_n$ is an intermediate point between θ_0 and $\hat{\theta}_n$.

According to (4.3), (4.4) and (4.2) we have the representation

$$\begin{aligned} u_n(t) &= W_n(t) - \dot{\Lambda}_0\left(t,\widetilde{\theta}_n\right)\left(-\frac{1}{\theta_0}I_0^{-1}\int_{-\infty}^{\infty} s\frac{\dot{\lambda}_0\left(\frac{s}{\theta_0}\right)}{\lambda_0\left(\frac{s}{\theta_0}\right)}\,dW_n\left(s\right) + O\left(n^{-1/4}\right)\right) \\ &= W_n(t) + \dot{\Lambda}_0\left(t,\widetilde{\theta}_n\right)I_0^{-1}\int_{-\infty}^{\infty}\frac{s}{\theta_0}\frac{\dot{\lambda}_0\left(\frac{s}{\theta_0}\right)}{\lambda_0\left(\frac{s}{\theta_0}\right)}\,dW_n\left(s\right) + O\left(n^{-1/4}\dot{\Lambda}_0\left(t,\widetilde{\theta}_n\right)\right). \end{aligned}$$

Furthermore, we put

$$\hat{u}_n(t) = W_n(t) + \dot{\Lambda}_0\left(t,\theta_0\right)I_0^{-1}\int_{-\infty}^{\infty}\frac{s}{\theta_0}\frac{\dot{\lambda}_0\left(\frac{s}{\theta_0}\right)}{\lambda_0\left(\frac{s}{\theta_0}\right)}\,dW_n\left(s\right). \tag{4.5}$$

We have

$$\dot{\Lambda}_0\left(t,\theta_0\right) = \Lambda_0\left(\frac{t}{\theta_0}\right) - \frac{t}{\theta_0}\lambda_0\left(\frac{t}{\theta_0}\right)$$

then

$$\hat{u}_n(t) = W_n(t) + \left(\Lambda_0\left(\frac{t}{\theta_0}\right) - \frac{t}{\theta_0}\lambda_0\left(\frac{t}{\theta_0}\right)\right)\int_{-\infty}^{\infty} L\left(\frac{s}{\theta_0}\right)\,dW_n(s)$$

where

$$L\left(\frac{s}{\theta_0}\right) = \mathrm{I}_0^{-1}\frac{s}{\theta_0}\frac{\dot{\lambda}_0\left(\frac{s}{\theta_0}\right)}{\lambda_0\left(\frac{s}{\theta_0}\right)}.$$

Introduce the stochastic process

$$
\begin{aligned}
\hat{u}(t) &= W\Big(\Lambda_0\left(t,\theta_0\right)\Big) \\
&\quad + \left(\Lambda_0\left(\frac{t}{\theta_0}\right) - \frac{t}{\theta_0}\lambda_0\left(\frac{t}{\theta_0}\right)\right)\int_{-\infty}^{\infty} L\left(\frac{s}{\theta_0}\right) dW\Big(\Lambda_0\left(s,\theta_0\right)\Big) \\
&= \sqrt{\theta_0}\Bigg[W\left(\Lambda_0\left(\frac{t}{\theta_0}\right)\right) \\
&\quad + \left(\Lambda_0\left(\frac{t}{\theta_0}\right) - \frac{t}{\theta_0}\lambda_0\left(\frac{t}{\theta_0}\right)\right)\int_{-\infty}^{\infty} L\left(\frac{s}{\theta_0}\right) dW\left(\Lambda_0\left(\frac{s}{\theta_0}\right)\right)\Bigg],
\end{aligned}
$$

where

$$W\left(at\right) = \sqrt{a}W(t).$$

It is easy to see that if we change the variables $\frac{t}{\theta_0} = u$ and $\frac{s}{\theta_0} = v$ in the integral; then we obtain the following equality

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \hat{u}\left(t\right)^2 \lambda\left(\frac{t}{\theta_0}\right) dt \\
&= \int_{-\infty}^{\infty} \theta_0\left[W\left(\Lambda_0\left(u\right)\right) + \left(\Lambda_0\left(u\right) - u\lambda_0\left(u\right)\right)\int_{-\infty}^{\infty} L\left(v\right) dW\left(\Lambda_0\left(v\right)\right)\right]^2 \lambda_0\left(u\right)\theta_0 du \\
&= \theta_0^2\int_{-\infty}^{\infty} \left[W\left(\Lambda_0\left(u\right)\right) + \left(\Lambda_0\left(u\right) - u\lambda_0\left(u\right)\right)\int_{-\infty}^{\infty} L\left(v\right) dW\left(\Lambda_0\left(v\right)\right)\right]^2 \lambda_0\left(u\right) du \\
&= \widetilde{\Delta}_0.
\end{aligned}
$$

Suppose that we already prove that

$$\widetilde{\Delta}_n \Longrightarrow \widetilde{\Delta}_0. \tag{4.6}$$

Therefore

$$\Delta_n = \frac{\widetilde{\Delta}_n}{\hat{\theta}_n^2} \Rightarrow \frac{\widetilde{\Delta}_0}{\theta_0^2} = \Delta_0.$$

Now our goal is to show the relation (4.6).

Let us show the convergence of one-dimensional distributions. The case of multi-dimensional distributions can be done in a similar but cumbersome way.

Lemma 18 *Let the conditions* **A** *be satisfied, then the finite dimensional distributions of the process* $\hat{u}_n(t), t \in \mathbb{R}$ *converge to the finite dimensional distributions of the process* $\hat{u}(t), t \in \mathbb{R}$ *as* $n \to \infty$.

Proof. The proof of this lemma is based on the Central Limit Theorem for stochastic integrals (see, e.g., Kutoyants [41, Theorem 1.1]). We follow the proof of this theorem. In Particular, we obtain the convergence when $n \to \infty$ of the characteristic function

$$\Phi_n(\mu) = \mathbf{E}_{\theta_0} \exp\{i\mu\, \hat{u}_n(t)\} = \mathbf{E}_{\theta_0} \exp\left\{ i\,\mu\, W_n(t) - i\,\mu\,\lambda_0\left(\frac{t}{\theta_0}\right)\hat{v}_n \right\}$$

to the characteristic function of the limit process

$$\Phi_0(\mu) = \mathbf{E}_{\theta_0} \exp\{i\mu\, \hat{u}(t)\}. \tag{4.7}$$

Indeed, we have

$$\begin{aligned} W_n(t) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[X_j(t) - \Lambda_0(t,\theta_0)\right] = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{-\infty}^{t} d\left[X_j(s) - \Lambda_0(s,\theta_0)\right] \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \mathbb{I}_{\{s<t\}} d\pi_j(s) \end{aligned} \tag{4.8}$$

where we put $\pi_j(t) = X_j(t) - \Lambda_0(t,\theta_0)$. On the other hand we have

$$\int_{-\infty}^{+\infty} L\left(\frac{s}{\theta_0}\right) dW_n(s) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{-\infty}^{+\infty} L\left(\frac{s}{\theta_0}\right) d\pi_j(s). \tag{4.9}$$

Taking into account the expressions (4.8) and (4.9), we obtain the following representation of $\hat{u}_n(t)$

$$\hat{u}_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{-\infty}^{+\infty} \left[\mathbb{I}_{\{s<t\}} + \dot{\Lambda}_0(t,\theta_0)\; L\left(\frac{s}{\theta_0}\right)\right] d\pi_j(s).$$

Thus we can calculate the characteristic function

$$\begin{aligned} \Phi_n(\mu) = \exp\Bigg\{ n \int_{-\infty}^{+\infty} \Bigg[&\exp\left\{ \frac{i\mu}{\sqrt{n}} \left[\mathbb{I}_{\{s<t\}} + \dot{\Lambda}_0(t,\theta_0)\; L\left(\frac{s}{\theta_0}\right)\right]\right\} \\ &-1 - \frac{i\mu}{\sqrt{n}} \left[\mathbb{I}_{\{s<t\}} + \dot{\Lambda}_0(t,\theta_0)\; L\left(\frac{s}{\theta_0}\right)\right]\Bigg] \Lambda_0(ds,\theta_0) \Bigg\}. \end{aligned}$$

By Taylor formula

$$e^{i\phi} - 1 - i\phi = \frac{(i\phi)^2}{2} + O(\phi^3)$$

when $(n \to +\infty)$, we obtain

$$\Phi_n(\mu) \to \exp\left\{-\frac{\mu^2}{2}\int_{-\infty}^{+\infty}\left[\mathbb{I}_{\{s<t\}} + \dot{\Lambda}_0(t,\theta_0)\; L\left(\frac{s}{\theta_0}\right)\right]^2 \lambda_0\left(\frac{s}{\theta_0}\right) ds\right\}. \qquad (4.10)$$

This expression (4.10) is equivalent to

$$\mathbf{E}_{\theta_0}\exp\left\{i\mu W\Big(\Lambda_0(t,\theta_0)\Big) + i\,\mu\,\dot{\Lambda}_0(t,\theta_0)\int_{-\infty}^{+\infty} L\left(\frac{s}{\theta_0}\right) dW\Big(\Lambda_0(s,\theta_0)\Big)\right\},$$

which is the characteristic function defined in (4.7).
Therefore, we have the convergence of the one-dimensional distributions. In the general case of the vector $(\hat{u}_n(t_1),\ldots,\hat{u}_n(t_k))$ we use the Wold device, i.e.; we consider the sum

$$S_n = \sum_{l=1}^{k} a_l \hat{u}_n(t_l)$$

and prove its asymptotic normality

$$S_n \Longrightarrow \mathcal{N}\left(0,\sigma^2\right), \qquad \sigma^2 = \sum_{l=1}^{k}\sum_{m=1}^{k} a_l a_m K(t_l,t_m)$$

where

$$K(t,s) = \Lambda(t\wedge s) + \lambda(t)\,\lambda(s)\int_{-\infty}^{\infty} h(s)^2\,\lambda(s)\,\mathrm{d}s$$
$$+\lambda(t)\int_{-\infty}^{s} L(s)\,\lambda(s)\,\mathrm{d}s + \lambda(s)\int_{-\infty}^{t} h(s)\,\lambda(s)\,\mathrm{d}s.$$

The detailed calculations follow directly the same way as for the one-dimensional proof.

Lemma 19 *Let the conditions* **A** *be satisfied, then for any $H>0$ with $|t|+|s|<H_1+H_2=H$, then*

$$\mathbf{E}_{\theta_0}\left|\hat{u}_n^2(t) - \hat{u}_n^2(s)\right|^2 \le C\left(1 + C_0 H^{7/2}\right)\sqrt{|t-s|} \qquad (4.11)$$

where $C = C(H) > 0$ and C_0 does not depend on n.

Proof. The Cauchy-Schwartz inequality allows to write

$$\mathbf{E}_{\theta_0}\left|\hat{u}_n^2(t) - \hat{u}_n^2(s)\right|^2 = \mathbf{E}_{\theta_0}\left|\hat{u}_n(t) - \hat{u}_n(s)\right|^2\;\left|\hat{u}_n(t) + \hat{u}_n(s)\right|^2 \le$$
$$\le \sqrt{\mathbf{E}_{\theta_0}\left|\hat{u}_n(t) - \hat{u}_n(s)\right|^4\,\mathbf{E}_{\theta_0}\left|\hat{u}_n(t) + \hat{u}_n(s)\right|^4}. \qquad (4.12)$$

First we consider the term

$$\mathbf{E}_{\theta_0}\left|\hat{u}_n(t)-\hat{u}_n(s)\right|^4.$$

Recall that the representation of $\hat{u}_n(t)$ defined in (4.5) can also be written as follows

$$\hat{u}_n(t)=\frac{1}{\sqrt{n}}\int_{-\infty}^{+\infty}\left[1\!\!\mathrm{I}_{\{s<t\}}+\dot{\Lambda}_0\left(t,\theta_0\right)\ L\left(\frac{s}{\theta_0}\right)\right]\ d\pi^{(n)}(s)$$

where we have put $\pi^{(n)}(t)=\sum_{j=1}^n\pi_j(t)$.
Then we have

$$\begin{aligned}&\mathbf{E}_{\theta_0}\left|\hat{u}_n(t)-\hat{u}_n(s)\right|^4\\ &=\ \mathbf{E}_{\theta_0}\left\{\frac{1}{\sqrt{n}}\int\left[1\!\!\mathrm{I}_{\{v<t\}}+\dot{\Lambda}_0\left(t,\theta_0\right)\ L\left(\frac{v}{\theta_0}\right)\right]\ d\pi^{(n)}(v)-\right.\\ &\qquad\left.-\frac{1}{\sqrt{n}}\int\left[1\!\!\mathrm{I}_{\{v<s\}}+\dot{\Lambda}_0\left(s,\theta_0\right)\ L\left(\frac{v}{\theta_0}\right)\right]\ d\pi^{(n)}(v)\right\}^4\\ &=\ \frac{1}{n^2}\mathbf{E}_{\theta_0}\left\{\int\ \left\{\left(1\!\!\mathrm{I}_{\{v<t\}}-1\!\!\mathrm{I}_{\{v<s\}}\right)+\right.\right.\\ &\qquad\left.\left.+\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]\ L\left(\frac{v}{\theta_0}\right)\right\}d\pi^{(n)}(v)\right\}^4\end{aligned}$$

Suppose that $s<t$. Let

$$\varphi(v)=1\!\!\mathrm{I}_{\{v<t\}}-1\!\!\mathrm{I}_{\{v<s\}}$$

then

$$\varphi(v):=\begin{cases}0 & s<t<v\\ 0 & v<s<t\\ 1 & s<v<t.\end{cases}$$

Hence

$$\begin{aligned}&\mathbf{E}_{\theta_0}\left|\hat{u}_n(t)-\hat{u}_n(s)\right|^4\\ &=\frac{1}{n^2}\mathbf{E}_{\theta_0}\left\{\int\!\!\left(1\!\!\mathrm{I}_{\{s<v<t\}}+\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]L\left(\frac{v}{\theta_0}\right)\right)d\pi^{(n)}(v)\right\}^4.\end{aligned}$$

This last equality can be evaluated as follows (we use here the Lemma 1.2 in [41] and put $\frac{v}{\theta_0} = w$).

$$
\begin{aligned}
&\frac{1}{n^2}\mathbf{E}_{\theta_0}\left\{\int\left(1\!\!{\rm I}_{\{s<v<t\}}+\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]L\left(\frac{v}{\theta_0}\right)\right)n\pi(dv)\right\}^4\\
&\leq \frac{C}{n^2}\int\left(1\!\!{\rm I}_{\{s<\theta_0 w<t\}}+\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]L(w)\right)^4 n\,\Lambda_0(dv,\theta_0)\\
&\;+\frac{C}{n^2}\left(\int\left\{1\!\!{\rm I}_{\{s<\theta_0 w<t\}}+\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]L(w)\right\}^2 n\,\Lambda_0(dv,\theta_0)\right)^2\\
&\leq \frac{C\theta_0}{n}\int_{\frac{s}{\theta_0}}^{\frac{t}{\theta_0}}\lambda_0(w)\,dw\\
&\qquad+C\theta_0^2\left(\int 1\!\!{\rm I}_{\{s<\theta_0 w<t\}}\lambda_0(w)\,dw\right)^2\\
&\qquad+\frac{C}{n^2}\int\left(\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]L(w)\right)^4 n\,\theta_0\lambda_0(w)dw\\
&\qquad+\frac{C}{n^2}\left(\int\left\{\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]L(w)\right\}^2 n\,\theta_0\lambda_0(w)dw\right)^2,
\end{aligned}
$$

where we have used

$$\mathbf{E}_{\theta_0}\left(\pi^{(n)}(v)\right)^2 = n\,\Lambda_0\left(v,\theta_0\right)$$

and the relation

$$d\left(\pi^{(n)}(v)\right) = \pi^{(n)}\left(dv\right) = n\,\pi\left(dv\right) = n\,\Lambda_0\left(dv,\theta_0\right)$$

holds.

If we put $\Delta_j = X_j(v) - \Lambda_0\left(v,\theta_0\right)$, then

$$
\begin{aligned}
\mathbf{E}_{\theta_0}\left(\pi^{(n)}(v)\right)^2 &= \mathbf{E}_{\theta_0}\left(\sum_{j=1}^{n}\left(\pi_j(v)\right)\right)^2\\
&= \mathbf{E}_{\theta_0}\left(\sum_{j=1}^{n}\left[X_j(v)-\Lambda_0\left(v,\theta_0\right)\right]\right)^2\\
&= \sum_{j=1}^{n}\sum_{i=1}^{n}\mathbb{E}_{\theta_0}\left(\Delta_j\Delta_i\right)\\
&= \sum_{j=1}^{n}\Lambda_0\left(v,\theta_0\right) = n\,\Lambda_0\left(v,\theta_0\right).
\end{aligned}
$$

Finally the following estimates

$$\frac{C\theta_0}{n}\int_{\frac{s}{\theta_0}}^{\frac{t}{\theta_0}}\lambda_0(w)\,dw \le \frac{K_1\theta_0}{n\theta_0}\left|t-s\right| = \frac{K_1}{n}\left|t-s\right|, \tag{4.13}$$

$$C\theta_0^2\left(\int_{-\infty}^{+\infty}\mathbb{I}_{\{s<\theta_0 w<t\}}\,\lambda(w)\,dw\right)^2 \le K_2\theta_0^2\frac{1}{\theta_0^2}\left|t-s\right|^2$$
$$= K_2\left|t-s\right|^2. \tag{4.14}$$

holds.

Further we have

$$\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)=\int_s^t\frac{\partial\dot{\Lambda}_0\left(u,\theta_0\right)}{\partial u}du.$$

$$\begin{aligned}\left(\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]L(w)\right)^4 &\le \left(\mid\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\mid\mid L(w)\mid\right)^4\\ &\le \left(\int_s^t\mid\frac{\partial\dot{\Lambda}_0\left(u,\theta_0\right)}{\partial u}\mid du\mid L(w)\mid\right)^4\\ &\le \left(\kappa\mid L(w)\mid\left|t-s\right|\right)^4.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{C}{n^2}\int_{-\infty}^{+\infty}\left(\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]L(w)\right)^4 n\,\lambda_0(w)\theta_0 dw\\ &\le\frac{C\theta_0}{n}\int_{-\infty}^{+\infty}\left[\left|t-s\right|\kappa\mid L(w)\mid\right]^4\lambda_0(w)\,dw\\ &\le\frac{C\theta_0}{n}\left|t-s\right|^4\int_{-\infty}^{+\infty}\left(\kappa\mid L(w)\mid\right)^4\lambda_0(w)\,dw\\ &\le\frac{C\theta_0}{n}\left|t-s\right|^4C_3\\ &\le\frac{K_3}{n}\left|t-s\right|^4.\end{aligned}\tag{4.15}$$

A similar calculation allows us to show also that

$$\begin{aligned}
&\frac{C}{n^2}\left(\int\left\{\left[\dot{\Lambda}_0\left(t,\theta_0\right)-\dot{\Lambda}_0\left(s,\theta_0\right)\right]L(w)\right\}^2 n\,\theta_0\lambda_0(w)dw\right)^2,\\
&\leq\ C\theta_0^2\left|t-s\right|^4\left(\int\left\{\kappa\mid L(w)\mid\right\}^2\lambda_0(w)dw\right)^2\\
&\leq C\theta_0^2\left|t-s\right|^4 C_4\\
&\leq K_4\left|t-s\right|^4. \qquad (4.16)
\end{aligned}$$

From the relations (4.13) - (4.16) we deduce

$$\mathbf{E}_{\theta_0}\left|\hat{u}_n(t)-\hat{u}_n(s)\right|^4\leq\frac{K_1}{n}\left|t-s\right|+K_2\left|t-s\right|^2+\frac{K_3}{n}\left|t-s\right|^4+\leq K_4\left|t-s\right|^4 \qquad (4.17)$$

For the second term of inequality (4.12), after a similar calculation, we obtain the following estimate

$$\mathbf{E}_{\theta_0}\left|\hat{u}_n(t)+\hat{u}_n(s)\right|^4\leq C\left(1+H^4\right). \qquad (4.18)$$

Now the estimate (4.11) holds from (4.17) and (4.18).

Lemmas 18 and 19 allow us, for any $H>0$ to establish the convergence in distribution

$$\int_{-H}^{H}\hat{u}_n^2(t)\lambda\left(\frac{t}{\theta_0}\right)dt\Longrightarrow\int_{-H}^{H}\hat{u}^2(t)\lambda\left(\frac{t}{\theta_0}\right)dt, \qquad (4.19)$$

(for the proof see [32], Theorem A.22).

Lemma 20 *Let the conditions* **A** *be fulfilled. Then for any $\varepsilon>0$ there exist $H>0$ and n_0 such that for all $n\geq n_0$, we have:*

$$\mathbf{P}_{\theta_0}\left(\int_{|s|>H}\hat{u}_n^2(s)\lambda\left(\frac{s}{\theta_0}\right)ds>\varepsilon\right)\leq\varepsilon. \qquad (4.20)$$

Proof. By Tchebychev's inequality we have:

$$\mathbf{P}_{\theta_0}\left(\int_{|s|>H}\hat{u}_n^2(s)\lambda\left(\frac{s}{\theta_0}\right)ds>\varepsilon\right)\leq\frac{1}{\varepsilon}\int_{|s|>H}\mathbb{E}_{\theta_0}\hat{u}_n^2(s)\lambda\left(\frac{s}{\theta_0}\right)ds.$$

Direct calculation allows to verify that

$$\sup_s\mathbf{E}_{\theta_0}\hat{u}_n^2(s)\leq C$$

where the constant $C>0$ does not depend on n. Hence

$$\int_{|s|>H}\mathbf{E}_{\theta_0}\hat{u}_n^2(s)\lambda\left(\frac{s}{\theta_0}\right)ds\leq C\int_{|s|>H}\lambda\left(\frac{s}{\theta_0}\right)ds\longrightarrow 0$$

as $H \to \infty$ because $\lambda\left(\frac{\cdot}{\theta_0}\right) \in \mathcal{L}_1$. This convergence allows us to say that for $n \geq n_0$ with some n_0 we obtain the estimate (4.20).

Therefore from (4.19) and (4.20) we have the convergence in distribution

$$\int_{-\infty}^{\infty} \hat{u}_n^2(t)\lambda\left(\frac{t}{\theta_0}\right) dt \Longrightarrow \int_{-\infty}^{\infty} \hat{u}^2(t)\lambda\left(\frac{t}{\theta_0}\right) dt. \tag{4.21}$$

The last step is the following Lemma.

Lemma 21 *Let the conditions* **A** *be fulfilled. Then*

$$\int \left(n\mathbf{E}_{\theta_0}\left[\widehat{\Lambda}_n(t) - \Lambda\left(t, \hat{\theta}_n\right)\right]^2 \left|\lambda\left(\frac{t}{\hat{\theta}_n}\right) - \lambda\left(\frac{t}{\theta_0}\right)\right|\right)^2 dt \xrightarrow[n\to\infty]{} 0. \tag{4.22}$$

Proof. We have

$$\mathbf{E}_{\theta_0}\left[\widehat{\Lambda}_n(t) - \Lambda\left(t, \hat{\theta}_n\right)\right]^2 \leq 2\mathbf{E}_{\theta_0}\left[\widehat{\Lambda}_n(t) - \Lambda\left(t, \theta_0\right)\right]^2 + 2\mathbf{E}_{\theta_0}\left[\Lambda\left(t, \theta_0\right) - \Lambda\left(t, \hat{\theta}_n\right)\right]^2.$$

It was shown above that

$$n^2 \sup_t \mathbf{E}_{\theta_0}\left[\widehat{\Lambda}_n(t) - \Lambda\left(\frac{t}{\theta_0}\right)\right]^4 < C,$$
$$n^2 \sup_t \mathbf{E}_{\theta_0}\left[\Lambda\left(\frac{t}{\theta_0}\right) - \Lambda\left(\frac{t}{\hat{\theta}_n}\right)\right]^4 \leq C.$$

Further

$$\begin{aligned}\left|\lambda\left(\frac{t}{\hat{\theta}_n}\right) - \lambda\left(\frac{t}{\theta_0}\right)\right| &= |t| \left|\frac{1}{\hat{\theta}_n} - \frac{1}{\theta_0}\right| \left|\dot{\lambda}\left(\frac{t}{\tilde{\theta}_n}\right)\right| \\ &= |t| \frac{\left|\hat{\theta}_n - \theta_0\right|}{\left|\hat{\theta}_n \times \theta_0\right|} \left|\dot{\lambda}\left(\frac{t}{\tilde{\theta}_n}\right)\right|.\end{aligned}$$

As we have the following convergence in probability $\hat{\theta}_n \to \theta_0$. Therefore we just have $\hat{\theta}_n = \theta_0 + r_n$ with $\lim_{n\to\infty} r_n = 0$.

$$\left|\lambda\left(\frac{t}{\hat{\theta}_n}\right) - \lambda\left(\frac{t}{\theta_0}\right)\right| = |t| \frac{\left|\hat{\theta}_n - \theta_0\right|}{|\theta_0^2 + \theta_0 \times r_n|} \left|\dot{\lambda}\left(\frac{t}{\tilde{\theta}_n}\right)\right|.$$

Now the convergence (4.22) holds from the Cauchy-Schwartz inequality

$$\left(\mathbf{E}_{\theta_0}\left[\widehat{\Lambda}_n(t)-\Lambda_0\left(\frac{t}{\theta_0}\right)\right]^2\left|\lambda\left(\frac{t}{\hat{\theta}_n}\right)-\lambda\left(\frac{t}{\theta_0}\right)\right|\right)^2$$
$$\leq \mathbf{E}_{\theta_0}\left[\widehat{\Lambda}_n(t)-\Lambda_0\left(\frac{t}{\theta_0}\right)\right]^4\mathbf{E}_{\theta_0}\left|\hat{\theta}_n-\theta_0\right|^2\times\left|\frac{t}{\theta_0^2+\theta_0 r_n}\dot{\lambda}\left(\frac{t}{\widetilde{\theta}_n}\right)\right|^2$$

and from the condition $\dot{\lambda}_0(\cdot)\in\mathcal{L}_1$.
Remember that $\mathbf{E}_{\theta_0}\left|\hat{\theta}_n-\theta_0\right|^2\to 0$.

4.3 Consistency of the Cramér-von Mises test with scale parameter

The following proposition gives us the consistency our test

Theorem 11 *The test*

$$\hat{\Psi}_n(X^{(n)})=1\!\!I_{\{\Delta_n>c_\epsilon\}}$$

is consistent under alternative $\mathscr{H}_1$*, that is, for any* $\Lambda\notin\mathcal{L}(\Theta)$ *we have:*

$$\beta\left(\hat{\Psi}_n,\Lambda\right)\xrightarrow[n\to\infty]{}1,$$

and it is uniformly consistent under alternatives ${\mathscr{H}_1}^\rho$*, that is,*

$$\inf_{\Lambda(\cdot)\in\mathcal{F}_\rho}\beta\left(\hat{\Psi}_n,\Lambda\right)\xrightarrow[n\to\infty]{}1.$$

Preuve : Under the hypothesis $\mathscr{H}_1$, the power $\beta\left(\hat{\Psi}_n,\Lambda\right)$ is

$$\begin{aligned}\beta\left(\hat{\Psi}_n,\Lambda\right) &= \mathbf{E}_\Lambda\left(\hat{\Psi}_n\right)=\mathbf{P}(\text{choose } \mathscr{H}_1|\mathscr{H}_1 \text{ is true})\\ &= \mathbf{P}\left(\Delta_n>c_\epsilon|\mathscr{H}_1\right)=\mathbf{P}_\Lambda(\Delta_n>c_\epsilon).\end{aligned}$$

We can write (as usual in such situation, see [21])

$$\begin{aligned}
\mathbf{P}_\Lambda\left(\Delta_n > c_\epsilon\right) &= \mathbf{P}_\Lambda\left(\sqrt{n}\left\|\hat\Lambda_n\left(\cdot\right)-\Lambda_0\left(\frac{\cdot}{\hat\theta_n}\right)\right\|_{\hat\theta_n} > \sqrt{c_\epsilon}\right)\\
&\geq \mathbf{P}_\Lambda\left(\sqrt{n}\left\|\Lambda\left(\cdot\right)-\Lambda_0\left(\frac{\cdot}{\hat\theta_n}\right)\right\|_{\hat\theta_n} - \sqrt{n}\left\|\Lambda(\cdot)-\hat\Lambda_n\left(\cdot\right)\right\|_{\hat\theta_n} > \sqrt{c_\epsilon}\right)\\
&= \mathbf{P}_\Lambda\left(\sqrt{n}\left\|\Lambda\left(\cdot\right)-\hat\Lambda_n\left(\cdot\right)\right\|_{\hat\theta_n} < \sqrt{n}\left\|\Lambda(\cdot)-\Lambda_0\left(\frac{\cdot}{\hat\theta_n}\right)\right\|_{\hat\theta_n} - \sqrt{c_\epsilon}\right)\\
&= 1-\mathbf{P}_\Lambda\left(\sqrt{n}\left\|\Lambda\left(\cdot\right)-\hat\Lambda_n\left(\cdot\right)\right\|_{\hat\theta_n} > \sqrt{n}\left\|\Lambda(\cdot)-\Lambda_0\left(\frac{\cdot}{\hat\theta_n}\right)\right\|_{\hat\theta_n} - \sqrt{c_\epsilon}\right)\\
&\geq 1-\mathbf{P}_\Lambda\left(\sqrt{n}\left\|\Lambda\left(\cdot\right)-\hat\Lambda_n\left(\cdot\right)\right\|_{\hat\theta_n} > \sqrt{n}g - \sqrt{c_\epsilon}\right),
\end{aligned}$$

where we used the properties of the norm

$$\|a-c\|-\|c-b\|\leq\|a-b\|$$

and put

$$g=\inf_{\theta\in\Theta}\left\|\Lambda(\cdot)-\Lambda_0\left(\frac{\cdot}{\theta_0}\right)\right\|_\theta > 0\,.$$

Thus

$$\mathbf{P}_\Lambda\left(\Delta_n > c_\epsilon\right) \quad\geq\quad 1-\frac{n\,\mathbf{E}_\Lambda\int_{-\infty}^{+\infty}\left(\hat\Lambda_n\left(t\right)-\Lambda(t)\right)^2\lambda_0\left(\frac{t}{\hat\theta_n}\right)\,dt}{\left(\sqrt{n}g-\sqrt{c_\epsilon}\,\right)^2}\,.$$

We can write

$$\begin{aligned}
&n\mathbf{E}_\Lambda\int_{-\infty}^{+\infty}\left(\hat\Lambda_n\left(t\right)-\Lambda(t))\right)^2\lambda_0\left(\frac{t}{\hat\theta_n}\right)\,dt\\
&\qquad= n\mathbf{E}_\Lambda\int_{-\infty}^{+\infty}\left(\hat\Lambda_n\left(s\hat\theta_n\right)-\Lambda(s\hat\theta_n)\right)^2\hat\theta_n\lambda_0(s)\,ds\\
&\qquad\leq \sup_s n\mathbf{E}_\Lambda\left(\hat\Lambda_n\left(s\hat\theta_n\right)-\Lambda(s\hat\theta_n)\right)^2\int_{-\infty}^{+\infty}\hat\theta_n\lambda_0(s)\,ds \leq C<\infty.
\end{aligned}$$

Hence

$$\mathbf{P}_\Lambda\left(\Delta_n > c_\epsilon\right)\geq 1-\frac{C}{\sqrt{n}g-\sqrt{c_\varepsilon}}\rightarrow 1.$$

Therefore the Cramér-von Mises type test is consistent for this alternative.

The proof presented above shows the uniform consistency of this test against the alternative $\mathscr{H}_1^{\rho}$.

Hence the theorem 11 has thus been proven.

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UNIVERSITE BRETAGNE LOIRE

Université du Maine

Thèse de Doctorat

Alioune TOP

Estimation Paramétriques et Tests d'Hypothèses pour des Modèles avec Plusieurs Ruptures d'un Processus de Poisson

Parametric Estimation and Hypothesis Testing for Models with Multiple Change-Point of Poisson Process

Résumé

Ce travail est consacré aux problèmes d'estimation paramétriques, aux tests d'hypothèses et aux tests d'ajustements pour les processus de Poisson non homogènes.

Tout d'abord on considère deux modèles ayant deux sauts localisés par un paramètre inconnu. Pour le premier modèle la somme des sauts est positive tandis que pour le second cette somme est nulle. Ainsi pour chaque modèle nous avons étudié les propriétés asymptotiques de l'estimateur bayésien (EB) et celui du maximum de vraisemblance (EMV). On a montré la consistance, la convergence en distribution et la convergence des moments. En particulier l'EB est asymptotiquement efficace. Pour le second modèle nous avons aussi considéré le test d'une hypothèse simple contre une alternative unilatérale et nous avons décrit les propriétés asymptotiques (choix du seuil et puissance) du test de Wald (WT) et du test du rapport de vraisemblance généralisé (GRLT).

Les démonstrations sont basées sur la méthode d'Ibragimov et Khasminskii. Cette méthode repose sur la convergence faible du rapport de vraisemblance normalisé dans l'espace de Skorohod sous certains critères de tension des familles de mesure correspondantes.

Par des simulations numériques, les variances limites nous ont permis de conclure que l'EB est meilleure que celui du EMV. Une approche numérique nous a permis aussi de trouver la valeur de l'EMV pour le modèle dont la somme des sauts est nulle.

Ensuite on a considéré le problème de construction d'un test d'ajustement pour un modèle avec un paramètre d'échelle. On a montré que dans ce cas, le test de Cramér-von Mises est asymptotiquement ''parameter-free'' et est consistent.

Mots clés

Estimation paramétrique, estimateur bayésien, estimateur du maximum de vraisemblance, processus de Poisson non homogènes, modèle de rupture, rapport de vraisemblance, test d'hypothèse.

Abstract

This work is devoted to the parametric estimation, hypothesis testing and goodness-of fit test problems for non homogenous Poisson processes.

First we consider two models having two jumps located by an unknown parameter. For the first model the sum of jumps is positive while the second is zero jumps sum. For each model, we studied the asymptotic properties of the Bayesian estimator (BE) and the maximum likelihood estimator (MLE).The consistency, the convergence in distribution and the convergence of moments are shown. In particular we show that the BE is asymptotically efficient. For the second model we also consider the problem of a simple hypothesis testing against a one- sided alternative. The asymptotic properties (choice of the threshold and power) of Wald test (WT) and the generalized likelihood ratio test (GRLT) are described.

For the proofs we use the method of Ibragimov and Khasminskii. This method is based on the weak convergence of the normalized likelihood ratio in the Skorohod space under some tightness criterion of the corresponding families of measure.

By numerical simulations, the limiting variances of estimators allows us to conclude that the BE outperforms the MLE. In the situation where the sum of jumps is zero, we developed a numerical approach to obtain the MLE.

Then we consider the problem of construction of goodness-of-test for a model with scale parameter. We show that the Cramér-von Mises type test is asymptotically parameter – free. It is also consistent.

Key Words

Parametric estimation, Bayesian estimator, maximum likelihood estimator, non homogenous Poisson process, change-point model, likelihood ratio, hypothesis testing.

L'Université Bretagne Loire